Pseudo-differential operators

Exercises 1 - 07.03.16

1. A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called *homogeneous os degree* $d \in \mathbb{R}$ if $f(rx) = r^d f(x)$, for all $r > 0$ and $x \neq 0$.

   (a) If f is continuous and homogeneous os degree $d \in \mathbb{R}$, show that there is a constant $C > 0$, depending on f, such that
   $$|f(x)| \leq |x|^d, \text{ for all} x \neq 0$$
   Determine the smallest C possible.

   (b) If f is k-times continuously differentiable and homogeneous os degree $d \in \mathbb{R}$, show that $\partial^\alpha f$ is homogeneous of degree $d - |\alpha|$, for all $|\alpha| \leq k$. Moreover, conclude that
   $$|\partial^\alpha f(x)| \leq C_\alpha |x|^{d-|\alpha|}, \text{ for all} x \neq 0.$$
   Here C_α depends on α and f, and $|\alpha| \leq k$.

2. Let $a > 0$. Compute the Fourier transformation of the functions $f_j : \mathbb{R} \to \mathbb{R}$:

   (a) $f_1(x) = e^{-ax}\chi_{[0,+\infty)}(x)$; (b) $f_2(x) = e^{-a|x|}$, (c) $f_3(x) = \chi_{[-a,a]}(x)$,

   Compare the properties of the functions f_j (continuity, differentiability, analyticity, and the decay for $|x| \to \infty$) with the corresponding properties of $\widehat{f_j}$.

3. Let $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. Prove that for any $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$ there s some $C_s, \alpha > 0$ such that
   $$|\partial_\xi^\alpha \langle \xi \rangle^s| \leq (1 + |\xi|)^{s-|\alpha|}, \text{ for all } \xi \in \mathbb{R}^n.$$
   Hint: the function $f(a, x) = (a_2 + |x|^2)^{m/2}$, where $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$, is homogeneous of degree m.

4. In the following, for $f \in \mathcal{S}$ and $m \in N$ let
   $$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta| \leq m} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial_x^\beta f(x) \right|.$$
   Prove that for every $\alpha \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$ there are constants $C_{m,\alpha}, C'_{m,\alpha} > 0$ such that
   $$|x^\alpha f|_{m,\mathcal{S}} \leq C_{m,\alpha} |f|_{m+|\alpha|,\mathcal{S}} \qquad \text{and} \qquad |\partial_x^\alpha f|_{m,\mathcal{S}} \leq C'_{m,\alpha} |f|_{m+|\alpha|,\mathcal{S}}$$
   uniformly in $f \in \mathcal{S}(\mathbb{R}^n)$.

5. Let $C^\infty_{\text{poly}}(\mathbb{R}^n)$ be the set of all smooth functions $m\colon \mathbb{R}^n \to \mathbb{C}$ of *polynomial growth*, i.e., for every $\alpha \in \mathbb{N}_0^n$ there exist a $k(\alpha) \in \mathbb{N}$ and $C_\alpha > 0$ with
   $$|\partial_x^\alpha m(x)| \leq C_\alpha (1 + |x|)^{k(\alpha)}, \qquad \text{for all } x \in \mathbb{R}^n.$$
   Moreover, let $m \in C^\infty_{\text{poly}}(\mathbb{R}^n)$ and let $(Mf)(x) := m(x)f(x)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

   (a) Prove that $M\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded operator, i.e., for all $k \in \mathbb{N}$ there exist $n(k) \in \mathbb{N}$ and $C > 0$ such that
   $$|Mf|_{k,\mathcal{S}} \leq C|f|_{n(k),\mathcal{S}}.$$

   (b) For any pair of functions $f, g \in \mathcal{S}(\mathbb{R}^n)$ the product fg lies in $\mathcal{S}(\mathbb{R}^n)$.

6. Let $(Mf)(x) := m(x)f(x)$ for $f \in \mathcal{S}(\mathbb{R}^n)$, where $m\colon \mathbb{R}^n \to \mathbb{C}$ is a smooth function. Prove that $m \in C^\infty_{\text{poly}}(\mathbb{R}^n)$ if $M\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded operator.

   Hint: First of all
   $$\sup_{x \in \mathbb{R}^n} |m(x)f(x)| \leq C|f|_{k,\mathcal{S}}, \qquad f \in \mathcal{S}(\mathbb{R}^n)$$
   for some $k \in \mathbb{N}$. Then consider $f(x) = (1 + |x|^2)^{-k/2} e^{-\varepsilon|x|^2/2}$, $\varepsilon > 0$.