Pseudo-differential operators

Exercises 3 - 04.04.16

1. Let $p \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and $q \in S^\ell_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ be pseudo-differential symbols, with $\ell, m \in \mathbb{R}$. Prove that $pq \in S^{m+\ell}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and that for every $k \in \mathbb{N}_0$ one has

$$|pq|^{(\ell+m)}_k \leq C_k |p|^{(m)}_k |q|^{(\ell)}_k,$$

   where C_k depends only on k and n.

2. Prove that if $p \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ then $D^\alpha_x D^\beta_\xi p \in S^{m-|\beta|}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, for all multi-indices α and β, and and that for every $k \in \mathbb{N}_0$ one has

$$|D^\alpha_x D^\beta_\xi p|^{(m-|\beta|)}_k \leq C |p|^{(m)}_{k+|\alpha|+|\beta|},$$

   where C depends only on α, β, k and n.

3. Let $p \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and $\varphi \in S(\mathbb{R}^n_\xi)$. Prove that the function

$$r(x, \varphi) \doteq p(x, \varphi) \cdot \varphi(\xi), \; x, \xi \in \mathbb{R}^n,$$

   is a symbol in $S^{-\infty}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

4. Prove that $p \in S^m_{1,0}$ if and only if $\langle \xi \rangle^{-m} p \in S^0_{1,0}$.

5. For each $G \in \mathsf{L}^\infty(\mathbb{R}^n)$ set

$$G(D_x) f \doteq \mathcal{F}^{-1}[G(\xi)\hat{f}(\xi)], \; \forall f \in L^2(\mathbb{R}^n). \tag{1}$$

   (a) Prove that $G(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$ and the mapping

$$\begin{aligned} \Phi \colon \mathsf{L}^\infty(\mathbb{R}^n) &\to \mathcal{L}(L^2(\mathbb{R}^n)) \\ G &\mapsto \Phi(G) \doteq G(D_x) \end{aligned}$$

   is linear and bounded. Moreover, show that for every $G, H \in \mathsf{L}^\infty(\mathbb{R}^n)$, we have

$$G(D_x) \circ H(D_x) = (G \cdot H)(D_x). \tag{2}$$

   (b) Prove that if $G \in C^\infty_{\text{poly}}(\mathbb{R}^n)$, then $G(D_x) \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ (defined similarly as in (1)), is a bounded operator, and for $G_j \in C^\infty_{\text{poly}}(\mathbb{R}^n)$, $j = 1, 2$, Equation (2) holds as well.

   (c) Let $p \in C^\infty_{\text{poly}}(\mathbb{R}^n)$. Prove that for all $\lambda \in \mathbb{C} \setminus \overline{p(\mathbb{R}^n)}$ we have $(\lambda - p(D_x))^{-1} \in \mathcal{L}(L^2(\mathbb{R}^n))$ and

$$(\lambda - p(D_x))(\lambda - p(D_x))^{-1} f = (\lambda - p(D_x))^{-1}(\lambda - p(D_x)) f = f$$

   for all $f \in \mathcal{S}(\mathbb{R}^n)$, where $p(D_x) f = \mathcal{F}^{-1}[p(\xi)\hat{f}(\xi)]$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

   (d) For which $\lambda \in \mathbb{C}$ there exists $(\lambda - \Delta)^{-1}$ in the sense above?