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Asymptotic Stability of Semigroups Associated to Linear *Weak* Dissipative Systems

A. F. PAZOTO AND J. C. VILA BRAVO
Institute of Mathematics, P.O. Box 68530
Federal University of Rio de Janeiro, UFRJ
21945-970 Rio de Janeiro, Brasil

J. E. MUÑOZ RIVERA
National Laboratory of Scientific Computations
LNCC/MCT, Rua Getúlio Vargas 333
Quitandinha, Petrópolis, CEP 25651-070, RJ-Brasil
and
Institute of Mathematics, UFRJ
21945-970 Rio de Janeiro, Brasil

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Abstract—In this work, we consider a class of linear dissipative evolution equations. We show that the solution of this class has a polynomial rate of decay as time tends to infinity, but does not have exponential decay.

Keywords—Dissipative systems, Decay rate, Semigroups.

1. INTRODUCTION

This paper is concerned with the stability of the C_0-semigroups associated with the following initial value problem:

$$Cu_{tt} + Au + Bu_t = 0, \tag{1.1}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{1.2}$$

where A, B, and C are self-adjoint positive definite operators with the domain $D(A) \subset D(C) \subset D(B)$ dense in a Hilbert space H.

We will show a class of operators A, B, and C, for which the above equation is dissipative but the corresponding semigroup is not exponentially stable. In addition, we show that the solution of equation (1.1) decays polynomially to zero as time goes to infinity. To do this we assume some spectral properties of the operators A, B, and C and proceed as in [1], where the authors present a systematic approach combining a theorem by Gearhart [2] (see also [3]) and Huang [4] in the theory of semigroups with partial differential equations techniques. To show the polynomial decay we use the energy method and a some perturbation arguments.

Such damping mechanisms, for which the corresponding system defines a semigroup which is not exponentially stable, we call weak dissipation.

The rest of this work is organized as follows. In Section 2, we show, under suitable hypotheses on the operators A, B, and C, the existence of solutions to equation (1.1). Finally, in Section 3, we show the asymptotic behaviour of the solution.

2. EXISTENCE OF SOLUTIONS

In this section, we use the semigroup approach to show existence and uniqueness of solution to (1.1),(1.2). To do that, we assume that C is a self-adjoint operator with inverse C^{-1}. Let us denote by $\tilde{A} = C^{-1}A$, $\tilde{B} = C^{-1}B$. Finally, we will assume that $\tilde{A}$ and $\tilde{B}$ are also self-adjoint positive definite operators. Then equations (1.1),(1.2) can be rewritten as

$$u_{tt} + \tilde{A}u + \tilde{B}u_t = 0, \tag{2.1}$$

$$u(0) = u_0, \qquad u_t(0) = u_1. \tag{2.2}$$

Let us denote

$$\mathcal{H} = \mathcal{D}\left(\tilde{A}^{1/2}\right) \times H.$$

Putting $v = u_t$, equation (2.1) can be written as the following initial value problem:

$$\begin{aligned} \frac{dU}{dt} &= A_B U, \\ U(0) &= U_0, \end{aligned} \tag{2.3}$$

with $U = (u, v)$, $U_0 = (u_0, u_1)$. Let us define

$$\mathcal{D}(A_B) = \left\{(u, v) \in \mathcal{D}\left(\tilde{A}\right) \times \mathcal{D}\left(\tilde{A}^{1/2}\right) : \tilde{A}u + \tilde{B}v \in H\right\} \tag{2.4}$$

and

$$A_B U = \begin{pmatrix} v \\ -\left(\tilde{A}u + \tilde{B}v\right) \end{pmatrix}. \tag{2.5}$$

Clearly, for $U \in \mathcal{D}(A_B)$,

$$\begin{aligned} (A_B U, U)_H &= \left(\tilde{A}^{1/2}v, \tilde{A}^{1/2}u\right)_H - \left(\tilde{A}u + \tilde{B}v, v\right)_H \\ &= \left(\tilde{A}^{1/2}v, \tilde{A}^{1/2}u\right)_H - \left(\tilde{A}^{1/2}u + \tilde{A}^{-1/2}\tilde{B}v, \tilde{A}^{1/2}v\right)_H \\ &= -\left\|\tilde{B}^{-1/2}v\right\|_H \leq 0. \end{aligned} \tag{2.6}$$

Thus, A_B is a dissipative operator. Under the above notation we can establish the following theorem.

Theorem 2.1. *Let us assume that $\tilde{A} = C^{-1}A$ and $\tilde{B} = C^{-1}B$ are self-adjoint operators with $\tilde{A} = C^{-1}A$ a positive definite and also a bijection operator between $D(\tilde{A})$ and $\mathcal{H}$. Then the operator A_B is the infinitesimal generator of a C_0-semigroup $S_B(t)$ of contraction in $\mathcal{H}$.*

Proof. Since $\mathcal{D}(\tilde{A})$ is dense in H and $\mathcal{D}(\tilde{A}) \times \mathcal{D}(\tilde{A}) \subset \mathcal{D}(A_B)$, then $\mathcal{D}(A_B)$ is dense in $\mathcal{H}$. Thus, to prove Theorem 2.1, it is sufficient to prove that $0 \in \rho(A_B)$. Let us take $F = (f, g) \in \mathcal{H}$, we will prove that there exists $Y = (u, v) \in \mathcal{A}_B$ satisfying

$$A_B Y = F, \tag{2.7}$$

i.e.,

$$\begin{aligned} v = f &\in \mathcal{D}\left(\tilde{A}^{1/2}\right), \\ -\tilde{A}u - \tilde{B}v = g &\in H. \end{aligned} \tag{2.8}$$

Taking $v = f$ obtained from (2.6), we get

$$-\tilde{A}u - \tilde{B}f = g \in H. \tag{2.9}$$

Since $\tilde{A}$ is invertible, we conclude that there exists u

$$u = -\tilde{A}^{-1}g + \tilde{A}^{-1}Bf. \tag{2.10}$$

Therefore, $Y = (u, v) \in \mathcal{D}(A_B)$ and satisfies (2.7). Thus, the proof is complete.

3. MAIN RESULT

Here we will use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [2] and Huang [4], independently (see also [3,5]).

THEOREM 3.1. *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(A) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \left\|(i\beta I - A)^{-1}\right\| < \infty$$

hold, where $\rho(A)$ is the resolvent set of A.

To study the asymptotic behaviour of the semigroup associated to (1.1), we assume that the operators A, B, and C have the same eigenvectors and the eigenvalues satisfy

$$\begin{aligned} &Aw_\nu = \lambda_\nu w_\nu, \\ &Bw_\nu = f(\lambda_\nu)w_\nu, \qquad \text{where } f(\lambda) = o(\lambda^{1-\beta-\alpha}), \quad 0 < \beta, \\ &Cw_\nu = g(\lambda_\nu)w_\nu, \qquad \text{where } g(\lambda) = o(\lambda^{1-\alpha}), \quad 0 \le \alpha, \\ &\qquad \lambda_\nu \to +\infty. \end{aligned} \tag{3.1}$$

The following theorem describes the main results of this paper.

THEOREM 3.2. *Let us suppose that hypothesis (3.1) holds. Let $S_B(t)$ be the C_0-semigroup of contractions generated by A_B, and*

$$E(t) = \frac{1}{2}\left\|A^{1/2}u\right\|_H^2 + \frac{1}{2}\left\|C^{1/2}u_t\right\|_H^2,$$

the energy associated to (1.1). Then, it follows that

(i) *$S_B(t)$ is not exponentially stable, but*
(ii) *there exists a positive constant c, such that*

$$E(t) \le \frac{c}{t}\, E_B(0), \qquad \forall\, t > 0,$$

where

$$E_B(t) = \frac{1}{2}\left\|L^{1/2}u_t\right\|_H^2 + \frac{1}{2}\left\|Q^{1/2}u\right\|_H^2, \qquad \forall\, t > 0,$$

with $L = CB^{-1}C$ and $Q = CB^{-1}A$.

Proof. To prove (i), we use Theorem 3.1. That is, let us take $F=(f,g)\in\mathcal{H}$ and let us denote by $U=(u,v)$ the solution of the system

$$i\lambda U - A_B U = F,$$

i.e.,

$$\begin{aligned} i\lambda u - v &= f,\\ i\lambda v + C^{-1}Au + C^{-1}Bv &= g. \end{aligned} \tag{3.2}$$

Let us take $f\equiv 0$ and $g=w_\nu$. We look for solution of the form $u=aw_\nu$ and $v=bw_\nu$, with $a,b\in\mathbb{C}$. From (3.2), we get that a and b satisfy

$$-\lambda^2 a w_\nu + \lambda_\nu g(\lambda_\nu)^{-1} a w_\nu + b f(\lambda_\nu) g(\lambda_\nu)^{-1} w_\nu = w_\nu.$$

Now, choosing $\lambda=\sqrt{\lambda_\nu g(\lambda_\nu)^{-1}}$, and using the above equation, we obtain

$$f(\lambda_\nu)g(\lambda_\nu)^{-1}w_\nu b = w_\nu \quad \Rightarrow \quad b=\frac{g(\lambda_\nu)}{f(\lambda_\nu)},$$

so we have

$$a=-\frac{g(\lambda_\nu)^{3/2}}{\lambda_\nu^{1/2}f(\lambda_\nu)}i. \tag{3.3}$$

Therefore, we have

$$u=-\frac{g(\lambda_\nu)^{3/2}}{\lambda_\nu^{1/2}f(\lambda_\nu)}iw_\nu \qquad \text{and} \qquad v=\frac{g(\lambda_\nu)}{f(\lambda_\nu)}w_\nu. \tag{3.4}$$

Now we claim that

$$\|U\|_{\mathcal{H}}\to+\infty, \qquad \text{as } \nu\to\infty.$$

In fact, using (3.4), we conclude that

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 = \left\|\tilde{A}^{1/2}u\right\|_H^2 + \|v\|_H^2 &= \left\|C^{-1/2}A^{1/2}\left(-\frac{g(\lambda_\nu)^{3/2}}{\lambda_\nu^{1/2}f(\lambda_\nu)}iw_\nu\right)\right\|_H^2 + \left\|\frac{g(\lambda_\nu)}{f(\lambda_\nu)}w_\nu\right\|_H^2 \\ &= \left(\frac{g(\lambda_\nu)}{f(\lambda_\nu)}\right)^2\|w_\nu\|_H^2 + \left(\frac{g(\lambda_\nu)}{f(\lambda_\nu)}\right)^2\|w_\nu\|_H^2 = o\left(\lambda_\nu^{2\beta}\right)\to+\infty. \end{aligned} \tag{3.5}$$

Recalling that

$$i\lambda U - A_B U = F \Longleftrightarrow U=(i\lambda I - A_B)^{-1}F,$$

it follows from (3.5) and Theorem 3.1 that $S_B(t)$ is not exponentially stable.

To prove (ii), we consider

$$Cu_{tt}+Au+Bu_t=0. \tag{3.6}$$

Multiplying by u_t and recalling the definition of E, we get that

$$\frac{d}{dt}E(t) = -\left\|B^{1/2}u_t\right\|_H^2. \tag{3.7}$$

On the other hand, applying the operator CB^{-1} to (3.6), we have

$$CB^{-1}Cu_{tt}+CB^{-1}Au+Cu_t=0. \tag{3.8}$$

Thus, multiplying equation (3.8) with u_t, we deduce that

$$\frac{dE_B}{dt} = -\left\|C^{1/2}u_t\right\|_H^2. \tag{3.9}$$

Multiplying equation (3.6) with u, we get

$$\frac{d\phi}{dt} = \left\|C^{1/2}u_t\right\|_H^2 - \left\|A^{1/2}u\right\|_H^2, \tag{3.10}$$

where

$$\phi(t) = (Cu, u_t)_H + \frac{1}{2}\left\|B^{1/2}u\right\|^2. \tag{3.11}$$

Consequently, from (3.9) and (3.10), we arrive to

$$\begin{aligned} \frac{d}{dt}\left\{E_B(t) + \varepsilon\phi(t)\right\} &= -\left\|C^{1/2}u_t\right\|_H^2 + \varepsilon\left\|C^{1/2}u_t\right\|_H^2 - \varepsilon\left\|A^{1/2}u\right\|_H^2 \\ &= -(1-\varepsilon)\left\|C^{1/2}u_t\right\|_H^2 - \varepsilon\left\|A^{1/2}u\right\|_H^2 \\ &\leq -\gamma_0 E(t), \end{aligned} \tag{3.12}$$

for some $\gamma_0 > 0$, since $0 < \varepsilon < 1$. Integrating (3.12) from 0 to t, we obtain

$$E_B(t) + \varepsilon\phi(t) + \gamma_0\int_0^t E(s)\,ds \leq E_B(0) + \varepsilon\phi(0), \qquad \forall\, t > 0,$$

which in particular implies that

$$\int_0^{+\infty} E(s)\,ds \leq cE_B(0), \tag{3.13}$$

for some positive constant c. Finally, we have

$$\frac{d}{dt}\{tE(t)\} = E(t) + t\frac{dE}{dt}(t) \leq E(t),$$

and from (3.7), we obtain after integrating the above identity that

$$tE(t) \leq \int_0^{+\infty} E(s)\,ds \leq cE_B(0) \quad \Rightarrow \quad E(t) \leq \frac{c}{t}E_B(0).$$

This completes the proof.

Application 1

Let us take $B = A^{-\beta}$, $C = I$ in (1.1). Then equation (1.1) can be written as

$$u_{tt} + Au + A^{-\beta}u_t = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1.$$

Choosing $0 < \beta$ and arguing as in Theorem 3.2, it can be proved that the corresponding semi-group $S_\alpha(t)$ is not exponentially stable, but its energy $E(t) = (1/2)\|A^{1/2}u\|^2 + (1/2)\|u_t\|^2$ has a polynomial decay. Note that $E_B(t) = (1/2)\|A^{(1+\beta)/2}u\|^2 + (1/2)\|A^{\beta/2}u_t\|^2$.

Application 2

Our result also can be applied to study mathematical models with frictional damping (see [6, p. 429])

$$\begin{aligned} u_{tt} - u_{xxtt} - u_{xx} + \gamma u_t &= 0, \\ u(0,t) = u(L,t) &= 0, \\ u(0) = u_0, \qquad u_t(0 &= u_1. \end{aligned}$$

Letting $C = I-(\cdot)_{xx}$ and $A = -(\cdot)_{xx}$, with H and $D(A)$ being $L^2(0,L)$ and $H_0^1(0,L)$, respectively, the above model may be written as equation (1.1). Hence, there is no exponential stability, but a polynomial decay as given in Theorem 3.2, where

$$E(t) = \frac{1}{2}\int_0^L |u_x|^2 + |u_t|^2 + |u_{xt}|^2\, dx, \qquad E_B(t) = \frac{1}{2}\int_0^L |u_{xx}|^2 + |u_t|^2 + |u_{xxt}|^2\, dx.$$

Similar results are obtained in the n-dimensional case. That is, let us take $\Omega \subset \mathbb{R}^2$ an open bounded set with smooth boundary. Consider

$$\begin{aligned} u_{tt} - \Delta u_{tt} - \Delta u + \gamma u_t = 0, &\qquad \text{in } \partial\Omega \times]0,\infty[,\\ u(x,t) = 0, &\qquad \text{on } \partial\Omega,\\ u(0) = u_0, \quad u_t(0) = u_1, &\qquad \text{in } \partial\Omega. \end{aligned}$$

Application 3

Let us denote by $\Omega \subset \mathbb{R}^2$ an open bounded set with smooth boundary. Let us consider the plate equation

$$\begin{aligned} u_{tt} - \Delta u_{tt} + \Delta^2 u + \gamma u_t = 0, &\qquad \text{in } \partial\Omega \times]0,\infty[,\\ u = \Delta u = 0, &\qquad \text{on } \partial\Omega,\\ u(0) = u_0, \quad u_t(0) = u_1, &\qquad \text{in } \partial\Omega. \end{aligned}$$

Letting $C = I - \Delta$ and $A = \Delta^2$, with H and $D(A)$ being $L^2(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$, respectively, the above model may be written as equation (1.1). Using Theorem 3.2, we conclude that there is no exponential stability, but a polynomial decay where

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u_t|^2 + |\Delta u|^2\, dx, \qquad E_B(t) = \frac{1}{2}\int_\Omega |u_t - \Delta u_t|^2 + |\Delta u|^2 + |\Delta^{3/2} u|^2\, dx.$$

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