Elementary Theory of Analytic Functions of One or Several Complex Variables

Dover Edition

Henri Cartan

Partial scrutiny,
Solutions of Selected Exercises,
Comments, Suggestions and Errata

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Concerning the first *Note*, P. 11, $O_k = \{S(X) \in K[[X]] \mid \omega(S) \geq k\}$ is a vector subspace of $K[[X]]$ since, for $\lambda_1, \lambda_2 \in K$ and

$$S_1(X) = \sum_{n \geq 0} a_{n,1} X^n, S_2(X) = \sum_{n \geq 0} a_{n,2} X^n \in O_k,$$

that is, $a_{n,1} = a_{n,2} = 0$ for each $n < k$, it follows that $\lambda_1 a_{n,1} + \lambda_2 a_{n,2} = 0$ for each $n < k$, that is,

$$\lambda_1 S_1(X) + \lambda_2 S_2(X) \in O_k.$$

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1.1.4, PP. 12-3

- (4.2)
  For the first relation, see the first half of P. 13.
- For the first sentence after (4.2), consider $T_1(Y) = T_2(Y) = T(Y)$. Hence

  $$S \circ (T_1 + T_2) = \sum_{n \geq 0} a_n (T_1(Y) + T_2(Y))^n = \sum_{n \geq 0} 2^n a_n (T(Y))^n$$

  and

  $$S \circ T_1 + S \circ T_2 = \sum_{n \geq 0} a_n (T_1(Y))^n + \sum_{n \geq 0} a_n (T_2(Y))^n = \sum_{n \geq 0} 2 a_n (T(Y))^n.$$

- The family $S_i \circ T$ is summable since $\omega(S_i \circ T) \geq \omega(S_i)$. In fact, let $\omega(S_i) = k$, that is, $a_{k,i} \neq 0$ and $a_{n,i} = 0$ for $n < k$. Then

  $$S_i(T(Y)) = a_{k,i} \left(b_1 Y + b_2 Y^2 + \cdots\right)^k + a_{k+1,i} (T(Y))^{k+1} + \cdots.$$

  Therefore $\omega(S_i \circ T) \geq k$.

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"... we can substitute U(X) for Y ...", *Proof*, PROP. 5.1, P. 14
Use (4.2), P. 12.

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(6.3), P. 15
Multiply each side of [1] $0 = \frac{dS}{dX} \frac{1}{S} + S \frac{d}{dX} \left(\frac{1}{S}\right)$ by $1/S$.

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PROP. 7.1, P. 15
By PROP. 5.1, P. 14, (7.2) can be rewritten as

$$S(X) \text{ does not have multiplicative inverse,} \qquad S'(X) \text{ has a multiplicative inverse.}$$

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1.2.1, P. 17
Concerning the completeness of **C** and the absolute convergence of $\sum_n u_n$, refer to *Elementos de Topologia Geral* of **Elon Lages Lima** (in Portuguese),[2] **Cor.**, p. *152*, and **EX. 9**, p. *154*.

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1.2.2, P. 18

- $||u + v|| \leq ||u|| + ||v||$ since $|(u+v)(x)| \leq |u(x)| + |v(x)| \leq ||u|| + ||v||$ for all $x \in E$.
- Normal convergence of $\sum_n u_n$ implies absolute convergence of $\sum_n u_n(x)$ by the comparison test for series; the converse is not true.[3]

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[1] See (6.2).

[2] Rio de Janeiro, SBM, 2009, ISBN 978-85-85818-43-2.

[3] Consider $u_n : \mathbf{R} \to \mathbf{R}$ such that u_0 is the zero function and, for all positive integer n, u_n is defined by

$$u_n(x) = \begin{cases} \frac{\sin \pi x}{n} & \text{for } x \in [n, n+1], \\ 0 & \text{for } x \notin [n, n+1]. \end{cases}$$

- $||v|| \leq \sum_n ||u_n||$ since $|v(x)| = \big|\sum_n u_n(x)\big| \leq \sum_n |u_n(x)| \leq \sum_n ||u_n||$ for all $x \in E$.
- **Errata/Comment**, ll. 16-7
  - P should be p.
  - Consider p is a nonnegative integer and $v, v_p : \mathrm{E} \to \mathrm{K}$ are functions with $||v||, ||v_p|| < +\infty$. By definition, to say that the sequence of terms v_p converges uniformly to v means that for each $\varepsilon > 0$ there is an index P such that

    $$|v(x) - v_p(x)| < \varepsilon \text{ for all indices } p \geq \mathrm{P} \text{ and all points } x \in \mathrm{E}.$$

    Hence, for $\lim_{p\to\infty} ||v - \sum_{n=0}^{p} u_n|| = 0$ to express that the partial sums $\sum_{n=0}^{p} u_n$ converge uniformly to v as p tends to infinity, write $v_p = \sum_{n=0}^{p} u_n$ for each p and note that

    $$|v(x) - \sum_{n=0}^{p} u_n(x)| \leq ||v - \sum_{n=0}^{p} u_n|| \text{ for all } x \in \mathrm{E}.$$

- Concerning the necessary and sufficient condition for the normal convergence of the series of functions $u'_n = u_n|\mathrm{A}$, suppose that $\sum_n ||u'_n|| < +\infty$ and denote $||u'_n|| = \varepsilon_n$ for each n. Hence, for each n, $|u_n(x)| \leq \varepsilon_n$ for all $x \in \mathrm{A}$ and $\sum_n \varepsilon_n < +\infty$. Now suppose that, for each n, there is some $\varepsilon_n \geq 0$ with $|u_n(x)| \leq \varepsilon_n$ for all $x \in \mathrm{A}$ and $\sum_n \varepsilon_n < +\infty$. Thus, since $||u'_n|| \leq \varepsilon_n$ for each n, $\sum_n ||u'_n|| \leq \sum_n \varepsilon_n < +\infty$.
- For "... the limit of a uniformly convergent sequence of *continuous* functions (on a topological space E) is continuous.", see the above-mentioned Lima's Book, **Prop. 11**, p. *120*.
- "*... the sum of a normally convergent series of continuous functions is continuous.*" since a normally convergent series is uniformly convergent.[4]

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1.2.3, PP. 19-21

- $\mathrm{I} = \{r \in [0, +\infty) \mid \sum_{n\geq 0} |a_n| r^n < +\infty\}$ is an interval since $0 \in \mathrm{I}$ and, if $r_0 \in \mathrm{I}$ and $0 \leq r \leq r_0$, then $\sum_{n\geq 0} |a_n| r^n \leq \sum_{n\geq 0} |a_n| r_0^n < +\infty$, that is, $r \in \mathrm{I}$.
- For an illustration of PROP. 3.1, see Figure 1.

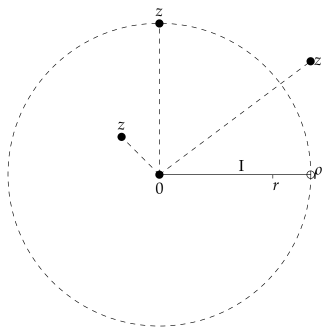

Figure 1: $\mathrm{S}(z)$ converges for $|z| < \rho$; it may converge or diverge for $|z| = \rho$; it diverges for $|z| > \rho$.

- Concerning PROP. 3.1 and ABEL'S LEMMA,

  $$\boxed{\text{"... the series } \textstyle\sum_{n\geq 0} a_n z^n \text{ converges normally for } |z| \leq r."}$$

  is an abuse of notation and language. By mathematical rigor, it can be rewritten as

  $$\boxed{\text{... the series } \textstyle\sum_{n\geq 0} u_n \text{ converges normally for } x \in \mathrm{A} = \{z \in \mathrm{K} \mid |z| \leq r\} \text{ with } u_n(x) = a_n x^n \text{ for each } x \in \mathrm{A}.}$$.[5]

[4] Apply the last item!
[5] See comment on $u_n|\mathrm{A}$, P. 18.

- For the two-line proof of ABEL'S LEMMA, let $u_n(z) = a_n z^n$ for $|z| \le r$. Now use the necessary and sufficient condition for the normal convergence of the series of functions u_n|A, P. 18.[6]

- The *Formula for the radius of convergence* is theoretical rather than practical.[7] In practice ρ is determined by the Ratio Test from
$$\mathrm{L} = \lim_{n\to\infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right|,$$
provided that the limit does not fail to exist.[8]

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1.2.4, PP. 21-2

- $\boxed{\text{“Thus both radii of convergence are } \ge \rho.\text{”}}$, P. 21, l. -2
Suppose $\rho\,(\mathrm{S}) < \rho$. Consider $\rho\,(\mathrm{S}) < \underline{r < \rho}$. The underlined inequality implies $r \le \rho\,(\mathrm{S})$!
(The same argument holds for $\rho\,(\mathrm{P})$.)

- $\boxed{\text{“The first is obvious, ...”}}$, P. 21, l. -1
Take the limit as $p \to \infty$ on both sides of
$$\sum_{n=0}^{p} c_n z^n = \sum_{n=0}^{p} a_n z^n + \sum_{n=0}^{p} b_n z^n.$$

- $\boxed{\sum_{n\ge 0} |w_n| \le \sum_{p\ge 0}\sum_{q\ge 0} |u_p|\cdot|v_q| = \alpha_0\beta_0}$, P. 22, l. 8
Take the limit as $p \to \infty$ on both sides of
$$\begin{aligned}\sum_{n=0}^{p} |w_n| &= |u_0v_0 + (u_0v_1 + u_1v_0) + \cdots + (u_0v_p + u_1v_{p-1} + \cdots + u_pv_0)| \\ &\le |u_0|\cdot|v_0| + |u_0|\cdot|v_1| + |u_1|\cdot|v_0| + \cdots + |u_0|\cdot|v_p| + |u_1|\cdot|v_{p-1}| + \cdots + |u_p|\cdot|v_0| \\ &= |u_0|\cdot(|v_0| + |v_1| + \cdots + |v_p|) + |u_1|\cdot(|v_0| + \cdots + |v_{p-1}|) + \cdots + |u_p|\cdot|v_0| \\ &\le |u_0|\cdot\left(\sum_{n=0}^{p}|v_n|\right) + |u_1|\cdot\left(\sum_{n=0}^{p}|v_n|\right) + \cdots + |u_p|\cdot\left(\sum_{n=0}^{p}|v_n|\right) \\ &= \left(\sum_{n=0}^{p}|u_n|\right)\cdot\left(\sum_{n=0}^{p}|w_n|\right).\end{aligned}$$

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Proof, PROP. 5.1, P. 23

- $\boxed{\text{“..., } \sum_{n\ge 1} |b_n| r^{n-1} \text{ is finite for sufficiently small } r > 0, \text{ ...”}}$, ll. 2-3
For a fixed $r > 0$, even if r is sufficiently small,
$$\frac{1}{r}\cdot\left(\lim_{p\to\infty}\sum_{n=1}^{p}|b_n|r^n\right) = \lim_{p\to\infty}\left(\frac{1}{r}\cdot\left(\sum_{n=1}^{p}|b_n|r^n\right)\right) = \lim_{p\to\infty}\sum_{n=1}^{p}|b_n|r^{n-1}.$$

- $\boxed{\text{“... } \sum_{n\ge 1}|b_n|r^n = r\cdot\left(\sum_{n\ge 1}|b_n|r^{n-1}\right) \text{ tends to 0 when } r \text{ tends to zero.”}}$, ll. 4-5
$f(r) = \sum_{n\ge 1}|b_n|r^{n-1}$ is bounded for sufficiently small $r > 0$ since otherwise $\lim_{r\to 0} f(r) = +\infty$![9]

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[6] See the last item.

[7] For more information on the limit superior, refer to the book **Real Analysis** of **H. L. Royden** and **P. M. Fitzpatrick** (Fourth Edition, Prentice Hall, 2010), **Prop. 19**, p. **23**. For example:

A sequence of real numbers $\{a_n\}$ converges to an extended real number a if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$.

[8] $|z| < \rho$ if $\mathrm{L} < 1$; $|z| > \rho$ if $\mathrm{L} > 1$; $|z| = \rho$ if $\mathrm{L} = 1$.

[9] See the last item!

- **Errata**, l. 8
  Replace $\mathrm{U}(x)$ by $\mathrm{U}(\mathrm{X})$.

- $\boxed{\text{“... } \mathrm{T} \to \mathrm{T}(z) \text{ is a ring homomorphism ...”}}$, l. 14
  It follows from PROP. 4.1, P. 21.

- $\boxed{\text{“... S converges at the point } \mathrm{T}(z)\text{, ...”}}$, l. 15
  In fact, since $|\mathrm{T}(z)| < \rho(\mathrm{S})$ for all z with $|z| \leq r$,[10] $\mathrm{S}(\mathrm{T}(z))$ is absolutely convergent.

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Proof, PROP. 6.1, P. 24

For $\mathrm{V}(\mathrm{Y}) = 1 + \sum_{n>0} \mathrm{Y}^n$ and $\mathrm{U}(\mathrm{X}) = 1 + \mathrm{S}(\mathrm{X})$, since $\rho(\mathrm{V})$ and $\rho(\mathrm{U})$ are $\neq 0$,[11] it follows that the radius of convergence of $\mathrm{T} = \mathrm{V} \circ \mathrm{U}$ is $\neq 0$.[12]

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Proof of prop. 7.1, P. 25

- $\boxed{\text{“..., there exists a finite M} > 0 \text{ such that } \alpha_n r'^n \leq \mathrm{M} \text{ for all } n\text{, ...”}}$, l. 8
  Take $\mathrm{M} = \sum_{n\geq 0} \alpha_n r'^n$.

- $\boxed{\text{“... the series } \sum_{n\geq 1} n \left(\frac{r}{r'}\right)^{n-1} \text{ converges, ...”}}$, l. 11
  It converges by the Ratio Test since

$$\mathrm{L} = \lim_{n\to\infty} \left| \frac{(n+1)(r/r')^n}{n(r/r')^{n-1}} \right| = \frac{r}{r'} < 1.$$

- $\boxed{\text{“... it follows that } \rho = \rho'\text{.”}}$, l. 13
  Suppose $\rho < \rho'$. Consider $\rho < \underline{r < \rho'}$. The underlined inequality implies $r \leq \rho$!
  (The same argument holds for $\rho' < \rho$.)

- (7.3)
  It follows from

$$\frac{\sum_{n\geq 0} a_n (z+h)^n - \sum_{n\geq 0} a_n z^n}{h} - \sum_{n\geq 0} n a_n z^{n-1} = \sum_{n\geq 1} a_n \left\{ \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right\}$$

  with

$$(z+h)^n - z^n = h\left\{ (z+h)^{n-1} + z(z+h)^{n-2} + \cdots + z^{n-2}(z+h) + z^{n-1} \right\}$$

  since, for all $a, b \in \mathrm{K}$,

$$a^n - b^n = (a-b)\left(a^{n-1} + b a^{n-2} + \cdots + b^{n-2} a + b^{n-1} \right).$$

  (The last equation holds for $b = 0$. For $b \neq 0$, consider

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1 \right),$$

  replace x by $\frac{a}{b}$ and multiply both sides by b^n.)

- $\boxed{\text{“Since } |z| \text{ and } |z+h| \text{ are } \leq r\text{, we have } |u_n(z,h)| \leq 2n\alpha_n r^{n-1}\text{; ...”}}$, l. -10
  It follows from

$$\begin{aligned} |u_n(z,h)| &\leq |a_n| \left\{ |z+h|^{n-1} + |z||z+h|^{n-2} + \cdots + |z|^{n-2}|z+h| + |z|^{n-1} + n|z|^{n-1} \right\} \\ &\leq \alpha_n \left\{ r^{n-1} + r r^{n-2} + \cdots + r^{n-2} r + r^{n-1} + n r^{n-1} \right\}. \end{aligned}$$

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[10] $|\mathrm{T}(z)| \leq \sum_{n\geq 1} |b_n||z|^n \leq \sum_{n\geq 1} |b_n| r^n < \rho(\mathrm{S})$.

[11] See *Some examples* and PROP. 4.1, P. 21, respectively.

[12] See PROP. 5.1, P. 22.

- $\boxed{\text{"... } \sum_{n\le n_0} u_n(z,h) \text{ is a polynomial in } h \text{ which vanishes when } h=0; \text{ ..."}}$, ll. -6 and -5
  It follows from
  $u_n(z,0) = a_n \left\{(z+0)^{n-1} + z(z+0)^{n-2} + \cdots + z^{n-2}(z+0) + z^{n-1} - nz^{n-1}\right\} = a_n \left\{nz^{n-1} - nz^{n-1}\right\} = 0.$

- $\boxed{\text{"... } \left|\sum_{n\le n_0} u_n(z,h)\right| \le \epsilon/2 \text{ when } |h| \text{ is smaller than a suitably chosen } \eta."}$, ll. -5 and -4
  Use that $f(h) = \sum_{n\le n_0} u_n(z,h)$ is continuous and $f(0) = 0$.[13]

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Errata, first line of 1.2.8, P. 26
Replace S(x) by S(X).

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1.3.1, P. 28

- (1.1) holds since

$$\frac{d}{dz}\left(\sum_{n\ge 0}\frac{1}{n!}z^n\right) \underbrace{=}_{a_n = \frac{1}{n!}, \text{ PROP. 7.1, P. 24}} \sum_{n\ge 0}\frac{n}{n!}z^{n-1} = \sum_{n\ge 1}\frac{1}{(n-1)!}z^{n-1} \underbrace{=}_{m=n-1} \sum_{m\ge 0}\frac{1}{m!}z^m.$$

- For w_n, use that

$$(z+z')^n = \sum_{0\le p\le n}\begin{pmatrix} n \\ p \end{pmatrix} z^p z'^{n-p}, \text{ with } \begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{p!(n-p)!}.$$

- The second equation (1.4) holds since

$$\frac{d}{dy}\left(\sum_{n\ge 0}\frac{i^n}{n!}y^n\right) \underbrace{=}_{a_n = \frac{i^n}{n!}, \text{ PROP. 7.1, P. 24}} \sum_{n\ge 0}\frac{ni^n}{n!}y^{n-1} = i\sum_{n\ge 1}\frac{1}{(n-1)!}(iy)^{n-1} \underbrace{=}_{m=n-1} i\sum_{m\ge 0}\frac{1}{m!}(iy)^m.$$

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$e^x = (e^{x/2})^2$, l. -2, P. 28
Set $z = z' = \frac{x}{2}$ in (1.2), P. 28.

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$\lim_{x\to-\infty} e^x = 0$, l. 4, P. 29
In fact, $\lim_{x\to-\infty} e^x \underbrace{=}_{(1.3),\text{ P. } 28} \lim_{x\to-\infty}\frac{1}{e^{-x}} \underbrace{=}_{t=-x} \lim_{t\to\infty}\frac{1}{e^t} = 0.$

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"We deduce that the function e^x of the real variable x increases strictly from 0 to $+\infty$.", ll. 5-6, P. 29
In fact, $\frac{d}{dx}(e^x) = e^x > 0$.[14]

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First line, 1.3.3, P. 30
$\overline{e^{iy}} = e^{-iy}$, that is,

$$\overline{1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \cdots} = 1 - iy - \frac{y^2}{2!} + \frac{iy^3}{3!} + \frac{y^4}{4!} - \frac{iy^5}{5!} + \cdots,$$

since: complex conjugation is a continuous function in **C**; the sum of a series is the limit of the sequence of its partial sums; the conjugate of a finite sum equals the sum of the conjugates of the summands.

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Proof, THEO., PP. 30-1

- $\boxed{\cos^2 y + \sin^2 y = 1}$, P. 30
  In fact, $|e^{iy}| = 1$.[15]

[13]See the last item.
[14]See the very end of the P. 28.
[15]See the fourth line of 1.3.3, P. 30.

- l. -14, P. 31
  Suppose $\cos y > 0$ for all $y \in \left[y_0, y_0 + \frac{1}{a}\cos y_0\right]$. Then $y_0 + \frac{1}{a}\cos y_0 \in \{y_1 \mid \cos y > 0 \text{ for all } y \in [y_0, y_1]\}$. Therefore [16] $y_0 + \frac{1}{a}\cos y_0 < y_0 + \frac{1}{a}\cos y_0$!

- l. -11, P. 31
  $\sin\frac{\pi}{2} = 1$ since $\cos\frac{\pi}{2} = 0$, $\cos^2\frac{\pi}{2} + \sin^2\frac{\pi}{2} = 1$ and $\sin\frac{\pi}{2} > 0$.

- For the sentence ending with the LEMMA, see the above-mentioned Lima's Book, **Prop. 5**, p. *180*.

- l. -4, P. 31
  $ie^{i\left(y-\frac{\pi}{2}\right)} = ie^{iy}e^{-i\frac{\pi}{2}} \overbrace{=}^{\text{First line, (1.3.3), P. 30}} ie^{iy}\overline{e^{i\frac{\pi}{2}}} = ie^{iy}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) = e^{iy}$.

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Comment, l. -1, P. 31
Concerning $\boxed{\text{“Analogous results can be deduced for the intervals } \underline{\left[\frac{\pi}{2}, \pi\right]}, \left[\pi, \frac{3\pi}{2}\right] \text{ and } \left[\frac{3\pi}{2}, 2\pi\right].\text{”}}$, the underlined part is missing.

==

1.3.4, PP. 32-3

- $2\pi\mathbf{Z}$ is the kernel of $\varphi : (\mathbf{R}, +) \ni y \to e^{iy} \in (\mathbf{U}, \cdot)$.

- For φ, consider The First Isomorphism Theorem for Groups.[17]

- Remark that each $u \in \mathbf{U}$ is uniquely associated with

$$\arg u = \varphi^{-1}(u) \in \{y + 2\pi\mathbf{Z} \mid y \in \mathbf{R}\}.^{18}$$

- For the second paragraph (the one before *General definition of argument*):
  - $2\pi\mathbf{Z}$ is compact since $p|I$ is continuous[19] and onto;
  - φ is a homeomorphism via **Prop. 5** on p. *180* of the above-mentioned Lima's Book.

- (4.1) holds since $\frac{t}{|t|} = e^{i\arg\left(\frac{t}{|t|}\right)}$.

- For the *Application*, $|t|^n e^{i\,n\arg t} = |a|e^{i\arg a}$.

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(5.1), P. 33
In fact, $e^x e^{iy} = |t|e^{i\arg t}$.

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Proof of PROP. 5.1, P. 34, ll. 1-9

- **Errata**, l. 1
  For $\boxed{\text{“Let us suppose \underline{the} that ...”}}$, remove the underlined word.

- $h(t)$ is continuous since $f(t)$ and $g(t)$ are both continuous by *Definition*.[20] (Furthermore, each one of the possible values of $\log t$ is continuous since $\log|t|$ and $\arg t$ are both continuous.)

- $h^{-1}(\{n\})$ is both open and closed for $n \in h(\mathrm{D}) \subset \mathbf{Z}$. In fact, on the one hand, $h^{-1}(\{n\})$ is closed since $\{n\}$ is closed and h is continuous. On the other hand, let N be an open neighbourhood of n such that $\mathrm{N} \cap \mathbf{Z} = \{n\}$. Then: $\mathrm{O} = h^{-1}(\mathrm{N})$ is open since N is open and h is continuous; $h^{-1}(\{n\}) \subset h^{-1}(\mathrm{N}) = \mathrm{O}$; $h^{-1}(\{n\}) \supset \mathrm{O}$ since $h(t)$ takes only integral values. Therefore $h^{-1}(\{n\}) = \mathrm{O}$ is open.

[16] See l. -15, P. 31.
[17] If $\varphi : \mathrm{G} \to \mathrm{H}$ is a surjective group homomorphism, then H is isomorphic to $\mathrm{G}/ker(\varphi)$.
[18] $y + 2\pi\mathbf{Z} = \{y, y \pm 2\pi, y \pm 4\pi, \ldots\}$.
[19] See the above-mentioned Lima's Book, **Prop. 4**, p. *179*.
[20] See P. 33.

- **Errata**, l. 7
  For $\boxed{\text{“Thus the set is empty or is equal \underline{to} D.”}}$, the underlined word is missing.

- For each integer k, $f(t) + 2k\pi i$ is a branch of $\log t$ since it is continuous and $e^{f(t)+2k\pi i} = e^{f(t)} \cdot 1 = t$.

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Example, P. 34
For the conclusion, see LEMMA, P. 31, and l. -1, P. 31.

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PROP. 6.1 and its proof, PP. 34-5

- $\mathrm{T}(u)$ is absolutely convergent for $|u| < 1$;[21] $\rho\,(\mathrm{T}) = 1$ is determined by the Ratio Test from

$$\mathrm{L} = \lim_{n\to\infty} \left| \frac{(-1)^n u^{n+1}/(n+1)}{(-1)^{n-1} u^n/n} \right| = |u|.$$

- For “To show that this is the principal branch, it is sufficient to verify that it takes the same value as the principal branch for a particular value of u,...”, use PROP. 5.1, P. 33.

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Proof of PROP. 6.2, P. 35

- **Errata**
  Right after the Newton’s Quotient, shouldn’t $\boxed{\text{“... } t \text{ tends to 0, ...”}}$ be $\boxed{\text{“... } h \text{ tends to 0, ...”}}$?

- Since $z = f(t)$ is continuous, h tends to 0 iff $l = f(t+h) - f(t)$ tends to 0. Thus

$$\begin{aligned}
\frac{d}{dt}(f(t)) &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \\
&= \lim_{h\to 0} \frac{f(t+h)-f(t)}{t+h-t} \\
&= \lim_{h\to 0} \frac{f(t+h)-f(t)}{e^{f(t+h)} - e^{f(t)}} \\
&= \lim_{h\to 0} \left(\frac{1}{\frac{e^{f(t+h)}-e^{f(t)}}{f(t+h)-f(t)}} \right) \\
&= \lim_{l\to 0} \frac{1}{\frac{e^{z+l}-e^z}{l}} \\
&= \frac{1}{\frac{d}{dz}(e^z)} \\
&= \frac{1}{e^z} \\
&= \frac{1}{e^{f(t)}} \\
&= \frac{1}{t}.
\end{aligned}$$

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1.4.1, PP. 36-7

- “The series S (X), if it exists, is *unique* by no. 8 of § 2.”, P. 36, second line right after *Def.* 1.1
  Suppose that there exists a $\mathrm{T}\,(\mathrm{X}) \in \mathrm{K}\,[[\mathrm{X}]]$ with $\rho\,(\mathrm{T}) \neq 0$ such that $f(x) = \mathrm{T}(x - x_0)$ for $|x - x_0|$ sufficiently small. Consider $z = x - x_0$. Hence $\mathrm{S}(z) = \mathrm{T}(z)$ for $|z|$ sufficiently small, which implies that $\mathrm{S} = \mathrm{T}$ by no. 8 of § 2, P. 26.

- **Errata**, P. 36, l. -17
  Shouldn’t $\boxed{\text{“... at } x\text{, ...”}}$ be $\boxed{\text{“... at } x_0\text{, ...”}}$?

[21] See PROP. 3.1.a), P. 19.

- "If D is open in **R**, D is a union of open intervals and, if D is also connected, D is an open interval.", P. 36, ll. -11 and -10
  Every nonempty open set is the disjoint union of a countable collection of open intervals.[22]

- *Def.* 1.2, P. 36
  S(X) depends on x_0!

- "..., if $f(x)$ is analytic in D, then $1/f(x)$ is analytic in the open set D excluding the set of points x_0 such that $f(x_0) = 0$.", P. 37, ll. 3-5
  Let $x_0 \in \mathrm{D}$ such that $f(x_0) \neq 0$. Then there exists some $\mathrm{S}(\mathrm{X}) = \sum_{n\geq 0} a_n \mathrm{X}^n$ whose radius of convergence is $\neq 0$ and which satisfies

$$f(x) = \sum_{n\geq 0} a_n (x - x_0)^n \qquad \text{for} \quad |x - x_0| \quad \text{sufficiently small.}$$

  Notice that $a_0 \neq 0$ since otherwise $f(x_0) = 0$. Hence, by PROP. 5.1 and PROP. 6.1, P. 14 and P. 24, there exists a unique $\mathrm{T}(\mathrm{X}) = \sum_{n\geq 0} b_n \mathrm{X}^n$ (whose radius of convergence is also $\neq 0$) such that $\mathrm{S}(\mathrm{X})\,\mathrm{T}(\mathrm{X}) = 1$. Therefore, by PROP. 4.1, P. 21, it follows that

$$\frac{1}{f(x)} = \sum_{n\geq 0} b_n (x - x_0)^n \qquad \text{for} \quad |x - x_0| \quad \text{sufficiently small.}$$

- A corollary of the first paragraph on P. 37
  If f and g are analytic in D, then f/g is analytic in D excluding each x_0 such that $g(x_0) = 0$.

- "Finally, proposition 5.1 of § 2 gives that, if f is analytic in D and takes its values in D′ and if g is analytic in D′, then the composed function $g \circ f$ is analytic in D.", P. 37, ll. 6-8
  Suppose that $y = f(x)$, $x_0 \in \mathrm{D}$ with $|x - x_0|$ small enough and $y_0 = f(x_0)$, which is in D′. Thus $|y - y_0|$ is sufficiently small since f is continuous. Furthermore, suppose that S (Y) and T (X) are formal series with $\rho(\mathrm{S}), \rho(\mathrm{T}) \neq 0$ such that $f(x) = \mathrm{T}(x - x_0)$ and $g(y) = \mathrm{S}(y - y_0)$. Therefore

$$\mathrm{T}(x - x_0) - \mathrm{T}(0) = f(x) - f(x_0) = y - y_0$$

  (with $|\mathrm{T}(x - x_0) - \mathrm{T}(0)|$ small enough) and

$$(g \circ f)(x) = g(f(x)) = g(y) = \mathrm{S}(y - y_0) = \mathrm{S}\left(\mathrm{T}(x - x_0) - \mathrm{T}(0)\right).$$

  Let us now consider two cases (such that, in any case, the result follows):

  - If $\mathrm{T}(0) = 0$, that is, $\omega(\mathrm{T}) \geq 1$, then

$$(g \circ f)(x) = \mathrm{S}\left(\mathrm{T}(x - x_0)\right) = (\mathrm{S} \circ \mathrm{T})(x - x_0)$$

    by PROP. 5.1, P. 22;
  - If $\mathrm{T}(0) \neq 0$, that is, $\omega(\mathrm{T}) = 0$, then consider

$$\mathrm{T}_0(\mathrm{X}) = \mathrm{T}(\mathrm{X}) - \mathrm{T}(0).$$

    Hence $\omega(\mathrm{T}_0) \geq 1$ and

$$(g \circ f)(x) = \mathrm{S}\left(\mathrm{T}_0(x - x_0)\right) = (\mathrm{S} \circ \mathrm{T}_0)(x - x_0)$$

    by PROP. 5.1, P. 22.

- *Examples of analytic functions*, P. 37
  Since, for each $x_0 \in \mathrm{D}$ and for each nonnegative integer k,

$$\mathrm{A}_k(x) = a_k = a_k \cdot (x - x_0)^0 + 0 \cdot (x - x_0)^1 + 0 \cdot (x - x_0)^2 + \cdots$$

  and

$$\mathrm{I}(x) = x = x_0 \cdot (x - x_0)^0 + 1 \cdot (x - x_0)^1 + 0 \cdot (x - x_0)^2 + 0 \cdot (x - x_0)^3 + \cdots$$

  are analytic in D, it follows that [23] $\mathrm{P} = \sum_{k=0}^{n} \mathrm{A}_k \mathrm{I}^k$ is analytic in D.[24]

[22] See the above-mentioned Royden-Fitzpatrick's Book, **Prop. 9**, p. **17**.
[23] See P. 37, first line.
[24] $\mathrm{P}(x) = \sum_{k=0}^{n} a_k x^k$.

==
1.4.2, P. 37
PROP. 2.1 is an immediate consequence of PROP. 2.2 by considering *Def.* 1.2 on P. 36 and $\rho(x_0) = \rho - |x_0|$.
==
Proof of proposition 2.2, P. 38

- l. 2
  $\mathrm{S}^{(p)}(x_0) = \sum_{n\geq p} n(n-1)\cdots(n-p+1)a_n(x_0)^{n-p} = \sum_{n\geq p} \frac{n(n-1)\cdots(n-p+1)[(n-p)!]}{(n-p)!}a_n(x_0)^{n-p}$, $n-p=q$ and $n(n-1)\cdots(n-p+1)[(n-p)!] = n!$ imply that $\mathrm{S}^{(p)}(x_0) = \sum_{q\geq 0} \frac{(p+q)!}{q!}a_{p+q}(x_0)^q$.

- **Errata**, l. 3
  Shouldn't $(x_0)^q$ be $(r_0)^q$?

- (2.3)
  The last $\leq$ comes from
  $$r^n = ((r-r_0)+r_0)^n = \sum_{0\leq p\leq n} \binom{n}{p} (r-r_0)^p(r_0)^{n-p}.$$
  (Note that each comma can be replaced by an $=$ sign and each one of the last two $\leq$ signs can be eliminated!)

- "The double series
  $$\sum_{p,q} \frac{(p+q)!}{p!q!}a_{p+q}(x_0)^q(x-x_0)^p$$
  is absolutely convergent by (2.3).", ll. 10-12
  In fact, since $|x-x_0| < \rho - r_0$, there exists $r_0 \leq r < \rho$ such that $|x-x_0| \leq r - r_0$ since, otherwise, if $|x-x_0| > r-r_0$ for each r such that $r_0 \leq r < \rho$, then
  $$|x-x_0| = \lim_{r\to\rho}|x-x_0| \geq \lim_{r\to\rho}(r-r_0) = \rho - r_0!$$

==
Note 1, PP. 38-39

- l. -1, P. 38
  For the first $=$, note that
  $$\frac{1}{1-ix}\cdot\left(1-i\frac{x-x_0}{1-ix_0}\right) = \frac{1}{1-ix}\cdot\left(\frac{1-ix_0-ix+ix_0}{1-ix_0}\right) = \frac{1}{1-ix_0}.$$
  For the second $=$, note that, if $z = \frac{x-x_0}{1-ix_0}$, then
  $$(1-iz)\cdot(1+iz+i^2z^2+i^3z^3+\cdots) = 1.$$
  For the power series expansion at x_0, pay attention to the uniqueness of (2.1), P. 37.[25]

- l. 1, P. 39
  $\boxed{\text{“This series converges for } |x-x_0| < \sqrt{1+(x_0)^2} \text{ ...”}}$ comes from the Ratio Test by considering
  $$\mathrm{L} = \lim_{n\to\infty}\left|\frac{i^{n+1}(x-x_0)^{n+1}/(1-ix_0)^{n+2}}{i^n(x-x_0)^n/(1-ix_0)^{n+1}}\right| < 1.$$

==
Note 2, P. 39
(2.4) follows from (2.3) since
$$\frac{1}{p!}\left|\mathrm{S}^{(p)}(x)\right|(r-r_0)^p \overbrace{\leq}^{-r_0 \leq -|x|} \frac{1}{p!}\left|\mathrm{S}^{(p)}(x)\right|(r-|x|)^p \leq \sum_{p\geq 0}\frac{1}{p!}\left|\mathrm{S}^{(p)}(x)\right|(r-|x|)^p \overbrace{\leq}^{(2.3)} \mathrm{A}(r) < +\infty.$$
==
Note 3, P. 39

[25]See the second line right after *Def.* 1.1, P. 36.

- 1st half
  $g'(x) = |x|$ if $g(x) = \begin{cases} x^2/2 & \text{for } x \geq 0; \\ -x^2/2 & \text{for } x < 0. \end{cases}$

- 2nd half
  Since $f^{(n)}(0) = 0$ for all integers $n \geq 0$,[26] if $f(x)$ were analytic, it would be identically zero in a neighbourhood of $x_0 = 0$ by the last THEO. on P. 39.

==

Indication of proof, P. 39

Consider the sufficient condition. By **The Lagrange Remainder Theorem**,[27] if n is a nonnegative integer, since $f : \mathrm{V} \to \mathbf{R}$ has $n+1$ derivatives, then for each $x \neq x_0$ in V, there is a point c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Therefore, for some $0 < r \leq \frac{1}{2t}$ such that $[x_0 - r, x_0 + r] \subset \mathrm{V}$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{for } |x - x_0| \leq r$$

since

$$\lim_{n \to \infty} [f(x) - p_n(x)] = 0.$$

---

[26] For example: $f^{(0)}(0) = f(0) = 0$;

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2}}{h} \\ &= \lim_{h \to 0} \left(h \cdot \frac{e^{-1/h^2}}{h^2} \right) \\ &= \left(\lim_{h \to 0} h \right) \cdot \left(\lim_{u \to \infty} \frac{u}{e^u} \right) \\ &= 0 \cdot \left(\lim_{u \to \infty} \frac{1}{e^u} \right) \\ &= 0 \cdot 0 \\ &= 0; \end{aligned}$$

$$\begin{aligned} f''(0) &= \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{2e^{-1/h^2}}{h^3}}{h} \\ &= 2 \cdot \left(\lim_{h \to 0} \frac{e^{-1/h^2}}{h^4} \right) \\ &= 2 \cdot \left(\lim_{u \to \infty} \frac{u^2}{e^u} \right) \\ &= 4 \cdot \left(\lim_{u \to \infty} \frac{u}{e^u} \right) \\ &= 4 \cdot \left(\lim_{u \to \infty} \frac{1}{e^u} \right) \\ &= 4 \cdot 0 \\ &= 0; \end{aligned}$$

and so on. (For $n > 0$, we have used L'hopital's Rule.)

[27] Refer to the book **Advanced Calculus** of **P. M. Fitzpatrick** (Second Edition, Thomson Brooks/Cole, 2006), **Theo. 8.8**, p. **203**.

In fact,

$$\begin{aligned}|f(x)-p_n(x)| &\le \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-x_0|^{n+1}\\ &\le \mathrm{M}t^{n+1}r^{n+1}\\ &\le \mathrm{M}\left(\frac{1}{2}\right)^{n+1}\end{aligned}$$

and $\lim_{n\to\infty}(1/2)^n = 0$.

==

1.4.3, *Proof* of THEO., P. 40

- For a) implies b), see ll. -2 and -1, P. 36; specifically, $a_n = \frac{1}{n!}\mathrm{S}^{(n)}(0) = \frac{1}{n!}f^{(n)}(x_0)$.[28]
- For b) implies c), see the above-mentioned Lima' s Book, pp. *81* (**Cor. 1**) and *87*.[29]

==

1.4.4, PP. 40-1

- P. 40, l. -1
  $a_0 = 0$.
- P. 41, l. 1
  $1 \le k = \omega(\mathrm{S})$ (where $\mathrm{S}(\mathrm{X}) \in \mathrm{K}[[\mathrm{X}]]$ and $f(x) = \mathrm{S}(x-x_0)$ for $|x-x_0|$ sufficiently small).
- P. 41, ll. 1-3
  The convergence takes place since

  $$\sum_{n\ge k} a_n(x-x_0)^{n-k} = \frac{1}{(x-x_0)^k}f(x)$$

  for sufficiently small $|x-x_0|>0$ and

  $$\sum_{n\ge k} a_n(x_0-x_0)^{n-k} = a_k.$$

- **Errata**, P. 41, l. 4
  Shouldn't $\boxed{\text{“... of } x.\text{”}}$ be $\boxed{\text{“... of } x_0.\text{”}}$?
- Last sentence
  In a compact space, every infinite subset has an accumulation point.[30]

==

Errata, P. 42

- l. -18
  $\boxed{\text{“If } f(x) \text{ analytic ...”}}$ should be $\boxed{\text{“If } f(x) \text{ is analytic ...”}}$.
- l. -15
  $\boxed{\text{“... defined and analytic an ...”}}$ should be $\boxed{\text{“... defined and analytic in ...”}}$.

---

[28] See (8.1), P. 26.

[29] Let X be a topological space and $S \subset X$.

- Let $S \ne \emptyset$ be both open and closed. Hence X is connected iff $S = X$.
- S is closed iff $\overline{S} \subset S$.

[30] See the above-mentioned Lima's Book, *p. 178*, **Prop. 1**.

==
Def., P. 42
Just for the sake of clarification, concerning a function $f(x)$, to be meromorphic in an open set D means that **there is** a discrete set $S \subset D$ consisting of poles of $f(x)$ such that $f(x)$ is defined and analytic in $D' = D - S$. (By the way, D' is clearly open.)
==
==
II
==
==
2.1.1, PP. 50-1

- P. 50
  Concerning the formula for change of variable in an ordinary integral for obtaining $\int_{\gamma_1} \omega = \pm \int_{\gamma} \omega$, note that
  $$\int_a^b f(t)\,dt = \int_{a_1}^{b_1} f(t(u))|t'(u)|\,du.$$

- **Errata**, P. 50, l. -1
  n should be $n+1$.

- **Errata**, P. 51, l. 2
  "A *piecewise differentiable path is defined to be a continuous mapping ...*" should be "A *piecewise differentiable path* is defined to be a continuous mapping ...".

- "The sum on the right hand side is independent of the decomposition.", P. 51, l. 8
  It follows from the fact that, here, a meaningful partition $a = t'_0 < t'_1 < t'_2 < \ldots < t'_{n'-1} < t'_{n'} < t'_{n'+1} = b$ of $[a,b]$ must be a refinement of $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n < t_{n+1} = b$. Therefore, for each $i \in \{1, \ldots, n+1\}$, there is an $i' \in \{1, \ldots, n'+1\}$ such that $t_{i-1} = t'_{i'-1}$, $t_i = t'_{i'+k(i)}$ and
  $$\int_{t_{i-1}}^{t_i} f(t)\,dt = \sum_{j=i'}^{i'+k(i)} \int_{t'_{j-1}}^{t'_j} f(t)\,dt.$$

- $\boxed{\text{A comment on the domain of } \gamma \text{ (to be considered right before the } \textit{Example}\text{, P. 51)}}$
  Without loss of generality, from now on, one may as well consider $a = 0$ and $b = 1$. As a matter of fact, a and b will denote points in D.[31]

- *Example*, P. 51
  Define:
  $[0, 1/4] \ni t \to \gamma_1(t) = (a_2, b_2 + (1-4t)(b_1 - b_2)) \in D$;
  $[1/4, 1/2] \ni t \to \gamma_2(t) = (a_1 + (2-4t)(a_2 - a_1), b_2) \in D$;
  $[1/2, 3/4] \ni t \to \gamma_3(t) = (a_1, b_1 + (3-4t)(b_2 - b_1)) \in D$;
  $[3/4, 1] \ni t \to \gamma_4(t) = (a_2 + (4-4t)(a_1 - a_2), b_1) \in D$;
  $$\gamma(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, 1/4], \\ \gamma_2(t) & \text{for } t \in [1/4, 1/2], \\ \gamma_3(t) & \text{for } t \in [1/2, 3/4], \\ \gamma_4(t) & \text{for } t \in [3/4, 1]. \end{cases}$$
  (The reason why I have considered $A \subset D$ is because of PROP. 2.2 and THE GREEN-RIEMANN FORMULA, PP. 53-4.)

==
Proof, LEMMA, P. 52
Concerning the closedness part of the proof, if $c \in D$ is in the closure of E, then c is the center of an open disk intersecting E.[32]
==
(2.1), P. 52

[31] See P. 52.
[32] See the previously-mentioned Lima's Book, p. *81*, l. 6.

- Now, by abuse of notation, a and b are the endpoints of γ.[33] Without loss of generality, one may as well consider $\gamma : [0,1] \to \mathrm{D}$ with $\gamma(0) = a$ and $\gamma(1) = b$.

- Since $d\mathrm{F} = \frac{\partial \mathrm{F}}{\partial x}\, dx + \frac{\partial \mathrm{F}}{\partial y}\, dy$ and $(\mathrm{F} \circ \gamma)'(t) = \frac{\partial \mathrm{F}}{\partial x}(\gamma(t))\, x'(t) + \frac{\partial \mathrm{F}}{\partial y}(\gamma(t))\, y'(t)$ (by the Chain Rule),[34] it follows that $\int_\gamma d\mathrm{F} = \int_0^1 (\mathrm{F} \circ \gamma)'(t)\, dt$.[35]

==

"As h tends to 0, the right hand side tends to $\mathrm{P}(x, y)$ because of the continuity of the function P.", P. 53, ll. $\pm$16 and -15

By **The Mean Value Theorem for Integrals**,[36] $\frac{1}{h}\int_x^{x+h} \mathrm{P}(\xi, y)\, d\xi = \mathrm{P}(\xi_0, y)$ for some point $\xi_0 \in [x, x+h]$.

==

Proof, THE GREEN-RIEMANN FORMULA, PP. 54-5

For a_1, a_2, b_1 and b_2, see *Example*, P. 51.

==

Proof, PROP. 3.1, P. 55

- For the sufficient condition, note that P and Q are continuous.[37]

- For the necessary condition:

  - "... $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle A contained in D with sides parallel to the axes, ..." since ω has a primitive in D.[38]

  - **Errata**, l. -1
    $\boxed{\text{"... 3.3 ..."}}$ should be $\boxed{\text{"... 3.1 ..."}}$.

==

On the first *necessary and sufficient condition*, PROP. 4.1, P. 56

Suppose $\int_\gamma \omega = 0$ whenever γ is the boundary of a small rectangle contained (with its interior) in D with sides parallel to the axes. Let $(x_0, y_0) \in \mathrm{D}$ and $\mathrm{D}_0 \subset \mathrm{D}$ be an open disk centered at (x_0, y_0). Hence ω has a primitive in D_0 by PROP. 2.2, P. 53. Conversely, suppose ω is closed and γ is the boundary of a small rectangle A contained (with its interior) in D with sides parallel to the axes. For every $n \in \mathbf{N}$, we may subdivide A into n^2 rectangles A_{ij}, $i, j = 1, \ldots, n$, similar to A, but of $1/n$ the length and width of A. Therefore, if γ_{ij} is the boundary of A_{ij}, it follows that

$$\int_\gamma \omega = \sum_{i=1}^n \sum_{j=1}^n \int_{\gamma_{ij}} \omega$$

since any edge of an A_{ij} inside A shows up twice, but with opposite orientation. Then, for n sufficiently large, every A_{ij} is contained in an open disk D_{ij} in which ω has a primitive, and thus $\int_{\gamma_{ij}} \omega = 0$ by PROP. 2.1, P. 52.

==

"We know from proposition 2.2 that any closed form in an *open disc* has a primitive.", P. 56, first sentence after PROP. 4.1

- By *Def.*, P. 56, for ω, closedness in an open set means local existence of primitives. Here, closedness in an open disc means global existence of primitives.

- For a proof, use a similar argument to the one that has just been used for working with the first *necessary and sufficient condition*, PROP. 4.1, P. 56.

==

On the closedness of ω, PROP. 4.2, P. 56

If $f(z)$ is a branch of $\log z$ in a neighbourhood D_0 of $z_0 \neq 0$ (see *Example*, P. 34), then $f(z)$ is a primitive of dz/z in D_0 by PROP. 6.2, P. 35, *Def.* and PROP. 2.1, P. 67, and (3.5) and (3.6), P. 68.

==

[33] See P. 49.
[34] See P. 49, ll. -8 and -1.
[35] See P. 50, l. 3.
[36] Refer to the previously-mentioned Fitzpatrick's Book, **Theo. 6.26**, p. **166**.
[37] See the definition of a *differential form*, P. 49.
[38] Use PROP. 2.1, P. 52.

Def., P. 57
F depends on τ, whereas f doesn't!
==
Proof, THEO. 1, PP. 57-8

- Uniqueness of f
  - "... the difference $F_1 - F_2$ of two primitives of ω is constant, ..."
    See the remarks preceding PROP. 2.1, P. 52.
  - Concerning the conclusion, consider $f_1 - f_2 = g$ and, without loss of generality, $I_0 = g^{-1}(u) \neq \emptyset$. On the one hand, I_0 is closed since $\{u\}$ is closed and g is continuous. On the other hand, I_0 is open since, if $\tau \in I_0$, then the neighborhood of τ in which g is constant is contained in I_0. Hence, by the connectedness hypothesis, $I_0 = I$.
- Existence of f
  - "Each point $\tau \in I$ has a neighbourhood (in I) mapped by γ into an open disc where ω has a primitive F."
    In fact, there is an open disk U_τ centered at $\gamma(\tau)$ in which ω has a primitive. Now consider $I_\tau = \gamma^{-1}(U_\tau)$.
  - "Since I is compact, we can find a finite sequence of points

    $$a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b,$$

    such that, for each integer i where $0 \leq i \leq n$, γ maps the segment $[t_i, t_{i+1}]$ into an open disc U_i in which ω has a primitive F_i."
    In fact, consider without loss of generality that all neighborhoods I_τ are open intervals. Since $I \subset \bigcup_{\tau \in I} I_\tau$, by the Borel-Lebesgue Theorem,[39] there are finitely many open intervals $I_{\tau_1}, \ldots, I_{\tau_m}$ such that $I \subset \bigcup_{j=1}^m I_{\tau_j}$. Then, taking into account the intersections of the open intervals I_{τ_j}, $\{[t_i, t_{i+1}]\}_{i=1}^n$ can be obtained from $\left\{I_{\tau_j}\right\}_{j=1}^m$.
  - "... $U_i \cap U_{i+1}$ ... is connected, so $F_{i+1} - F_i$ is constant in $U_i \cap U_{i+1}$."
    See the remarks preceding PROP. 2.1, P. 52.

==
Proof, PROP. 5.1, P. 59
For the conclusion, use PROP. 5.1, P. 33.
==
HOMOTOPY, PP. 59-61

- *Def.*, P. 59
  - For an illustration, see Figure 2.
  - Note that (6.2) does not guarantee γ_0 and γ_u, $u \in (0, 1]$, with fixed end points.
  - γ_0 and γ_1 are continuous by (6.1-2), and, since ω is closed, the hypothesis of differentiability of γ_0 and γ_1 is not necessary for THEO. 2-2′, P. 60.[40]
- *Def.*, P. 60
  For fixed $v = u$, $f(t, u)$ is a primitive of ω along γ_u where $\gamma_u(t) = \delta(t, u)$.[41] Therefore the conclusion of *Proof of theorem* 2, P. 61, follows from the *Note* on P. 58.
- **Errata**, LEMMA, P. 60
  Replace the comma by a period.

==
2.1.7, PP. 61-2

[39] See the previously-mentioned Lima's Book, p. *174*, ll. -16, -15 and -14.
[40] See *Note*, P. 58.
[41] Compare (P) with (P′).

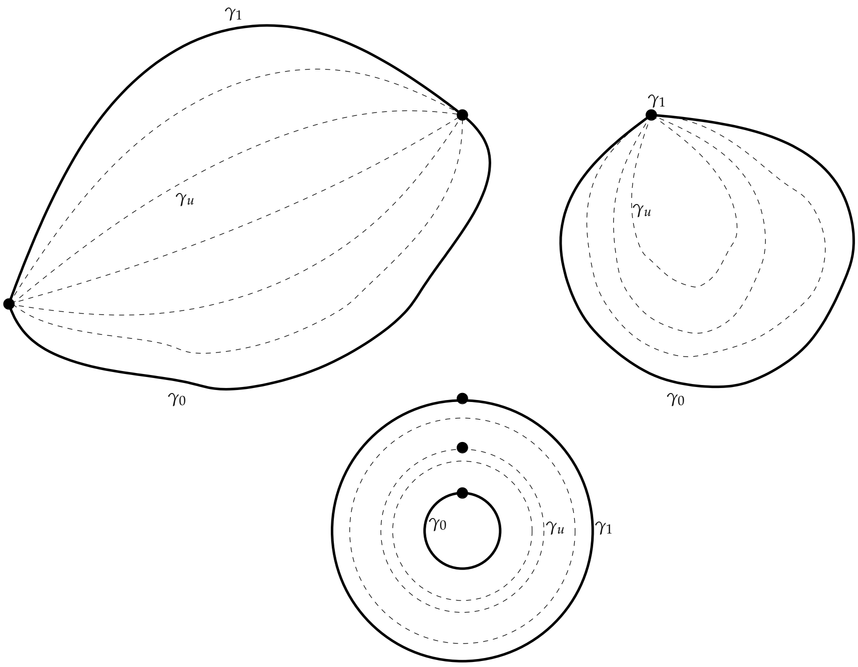

Figure 2: Homotopic paths.

- Simply connectedness transforms a local property, the existence of primitives of ω, into a global one.
- E is a starred subset of the plane with respect to $a \in \mathrm{E}$ iff

$$\{a + u(z - a) \ : \ u \in [0, 1]\} \subset \mathrm{E}$$

  for any $z \in \mathrm{E}$. (See Figure 3.)

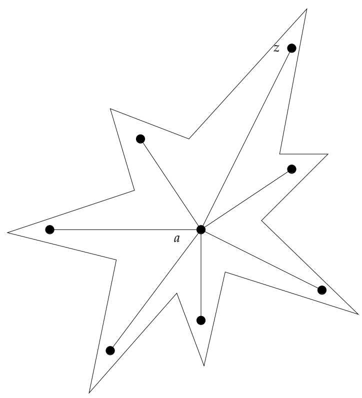

Figure 3: Starred subset of the plane with respect to a.

- First paragraph, P. 62
  Let D be open and starred with respect to a. On the one hand, D is path-connected, that is, for any two points in D, there is a path starting at one point and ending at the other. Then, since every path-connected topological space is connected,[42] D is connected. On the other hand, if $u \in [0, 1]$, then the homothety $z \to a + u(z - a)$ continuously transforms D into itself. Now, if γ is a closed path such that $\gamma(0) = a = \gamma(1)$, then $\delta(t, u) = a + (1 - u)(\gamma(t) - a)$ makes γ be homotopic to a.[43] Therefore, by *Def.*, P. 61, D is simply connected.

[42] See the previously-mentioned Lima's Book, p. *90*, **Prop. 27**.
[43] See (6.2), P. 59.

- $d) \Leftrightarrow b)$
  Let $\mathrm{I} = [0,1]$. Suppose first that $d)$ holds and ϕ is a continuous mapping of the circle $|z| = 1$ into D. Define paths $\gamma_0, \gamma_1 : \mathrm{I} \to \mathrm{D}$ by

  $$\gamma_0(t) = \phi(\cos(\pi(1-t)), \sin(\pi(1-t))) \text{ and } \gamma_1(t) = \phi(\cos(-\pi(1-t)), \sin(-\pi(1-t)))$$

  for $t \in \mathrm{I}$. Hence $\gamma_0(0) = \gamma_1(0) = \phi(-1,0)$ and $\gamma_0(1) = \gamma_1(1) = \phi(1,0)$. By $d)$, γ_0 and γ_1 are homotopic with fixed end points via some path-homotopy $(t,u) \to \delta(t,u)$ of $\mathrm{I} \times \mathrm{I}$ into D. Now, since the onto map

  $$(t,u) \to \varphi(t,u) = (\cos(\pi(1-t)), \sin(\pi(1-t))\sin(\pi(1/2-u)))$$

  of $\mathrm{I} \times \mathrm{I}$ into the disk $|z| \leq 1$ is such that $\varphi(\{0\} \times \mathrm{I}) = \{(-1,0)\}$ and $\varphi(\{1\} \times \mathrm{I}) = \{(1,0)\}$, and δ is constant on $\varphi^{-1}(\{(-1,0)\})$ and $\varphi^{-1}(\{(1,0)\})$, there is an induced map $\bar{\delta}$ of the disk $|z| \leq 1$ into D that extends ϕ.[44] In fact, $\bar{\delta}(\cos(\pi(1-t)), \sin(\pi(1-t))) = \bar{\delta} \circ \varphi(t,0) = \delta(t,0) = \gamma_0(t) = \phi(\cos(\pi(1-t)), \sin(\pi(1-t)))$ and $\bar{\delta}(\cos(-\pi(1-t)), \sin(-\pi(1-t))) = \bar{\delta} \circ \varphi(t,1) = \delta(t,1) = \gamma_1(t) = \phi(\cos(-\pi(1-t)), \sin(-\pi(1-t)))$. Conversely, suppose that $b)$ holds and $\gamma_0, \gamma_1 : \mathrm{I} \to \mathrm{D}$ are paths with the same end points. Consider the mapping ϕ of the circle $|z| = 1$ into D defined by $\phi(\cos(\pi(1-t)), \sin(\pi(1-t))) = \gamma_0(t)$ and $\phi(\cos(-\pi(1-t)), \sin(-\pi(1-t))) = \gamma_1(t)$ for $t \in \mathrm{I}$; this is well-defined since $\phi(\cos(\pi(1-0)), \sin(\pi(1-0))) = \gamma_0(0) = \gamma_1(0) = \phi(\cos(-\pi(1-0)), \sin(-\pi(1-0)))$ and $\phi(\cos(\pi(1-1)), \sin(\pi(1-1))) = \gamma_0(1) = \gamma_1(1) = \phi(\cos(-\pi(1-1)), \sin(-\pi(1-1)))$ (and is continuous by gluing on locally finite closed covers). By $b)$, there is a map ψ of the disk $|z| \leq 1$ into D which is an extension of ϕ . For the quotient map φ given above, it is straightforward to check that $\psi \circ \varphi$ is a path-homotopy between γ_0 and γ_1.

==
2.1.8, PP. 62-4

- $\mathrm{I}(\gamma, a)$ is also known as the *winding number* of γ around a; it represents the total number of times that $\gamma(t)$ travels counterclockwise around a and is negative if $\gamma(t)$ travels around a clockwise.

- The lines between (8.1) and PROPERTIES OF THE INDEX
  On the one hand, consider $z \to g_a(z) = z - a$. Then $g_a \circ \gamma$ is a closed path in $\mathbf{C}$, 0 does not belong to the image of $g_a \circ \gamma$ and

  $$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \underbrace{=}_{u = g_a(z)} \frac{1}{2\pi i}\int_{g_a \circ \gamma} \frac{du}{u} \in \mathbf{Z}.$$ [45]

  On the other hand, assume (without loss of generality) that $[0,1]$ is the domain of γ. Therefore, for any $\tau \in [0,1]$, there exists a primitive F of du/u in a neighbourhood of the point $(g_a \circ \gamma)(\tau) \in \mathbf{C} - \{0\}$ such that $e^{f(t)} = e^{\mathrm{F}((g_a \circ \gamma)(t))} = (g_a \circ \gamma)(t)$ for t near enough to τ;[46] $f(t)$ is a primitive of $\frac{du}{u}$ along $g_a \circ \gamma$. Now use (5.2), P. 58.

- 1)
  Denote $\gamma = \gamma_0$. By abuse of language, suppose that

  $$[0,1] \ni t \to \gamma_t \in \mathrm{C}^0([0,1], \mathbf{C} - \{a\})$$

[44] For the existence of a map $\bar{\delta}$ for which the diagram

$$\begin{array}{ccc} \mathrm{I} \times \mathrm{I} & \xrightarrow{\delta} & \mathrm{D} \\ \varphi \searrow & & \nearrow \bar{\delta} \\ & \text{closed unit disk} & \end{array}$$

commutes, see the remarks on co-induced topology and quotient space of the previously-mentioned Lima's Book, pp. *67-9*. Hence, if $\sim$ is the equivalence relation on $\mathrm{I} \times \mathrm{I}$ defined by δ (via (6.1), P. 59), that is,

$$(t,u) \sim (t',u') \Leftrightarrow \delta(t,u) = \delta(t',u'),$$

and $\varphi : \mathrm{I} \times \mathrm{I} \to (\mathrm{I} \times \mathrm{I})/\sim$ is the quotient map, then $(\mathrm{I} \times \mathrm{I})/\sim$ represents the closed unit disk.

[45] See PROP. 5.1, P. 58.

[46] See:

- (the first two sentences following) PROP. 4.2, P. 56;
- *Def.* and THEO. 1, P. 57;
- *Def.*, P. 33.

is a continuous deformation between γ_0 and γ_1 such that, for each $t \in [0,1]$, γ_t is closed and a does not belong to the image of γ_t. Hence

$$\mathrm{I}(\gamma_0, a) = \frac{1}{2\pi i}\int_{\gamma_0}\frac{dz}{z-a} = \frac{1}{2\pi i}\int_{\gamma_1}\frac{dz}{z-a} = \mathrm{I}(\gamma_1, a).$$

- 2)
First proof (easy)
For $|a-a'|$ small enough if necessary, a suitable traslation moves $g_a \circ \gamma$ to $g_{a'} \circ \gamma$.[47] Then

$$\mathrm{I}(\gamma, a) = \frac{1}{2\pi i}\int_{g_a\circ\gamma}\frac{du}{u} = \frac{1}{2\pi i}\int_{g_{a'}\circ\gamma}\frac{du}{u} = \mathrm{I}(\gamma, a').$$

Second proof (difficult)
Consider that $\gamma_{b,b'}$ is a parametrization of the line segment $\{b + t(b'-b) \mid t \in [0,1]\}$ with endpoints $b = (g_a \circ \gamma)(0) = (g_a \circ \gamma)(1)$ and $b' = (g_{a'} \circ \gamma)(0) = (g_{a'} \circ \gamma)(1)$. (See Figure 4.) Hence, for $|a-a'|$ small

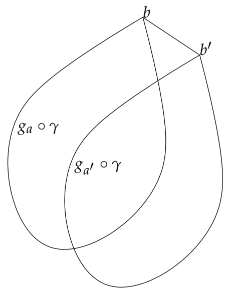

Figure 4: Homotopy between the juxtaposition $\gamma_{b,b'} * (g_a \circ \gamma) * \gamma_{b',b}$ and $g_{a'} \circ \gamma$.

enough if necessary,

$$\begin{aligned}
\mathrm{I}(\gamma, a) &= \frac{1}{2\pi i}\int_{g_a\circ\gamma}\frac{du}{u} \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_{b,b'}}\frac{du}{u} + \int_{g_a\circ\gamma}\frac{du}{u} + \int_{\gamma_{b',b}}\frac{du}{u}\right) \\
&= \frac{1}{2\pi i}\int_{\gamma_{b,b'}*(g_a\circ\gamma)*\gamma_{b',b}}\frac{du}{u} \\
&= \frac{1}{2\pi i}\int_{g_{a'}\circ\gamma}\frac{du}{u} \\
&= \mathrm{I}(\gamma, a'),
\end{aligned}$$

where $*$ represents juxtaposition of paths.

- 4)
By (4.1), P. 56, $\mathrm{I}(\gamma, 0) = +1$ for each circle γ of center $c = 0$ and radius $r > 0$ ($z = re^{it}$ with t running from 0 to 2π). If γ is not centered at the origin ($z = c + re^{it}$ with t running from 0 to 2π) and $0 < |c| < r$ (0 is inside the circle), then

$$\mathrm{I}(\gamma, 0) \overbrace{=}^{2)} \mathrm{I}(\gamma, c) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-c} = \frac{1}{2\pi i}\int_0^{2\pi} i\,dt = +1.$$

- **Errata**, P. 63, l. -3
Concerning $\boxed{\text{“... two } f \text{ continuous ...”}}$, remove f.

[47] See the item before the last one!

- Proof of PROP. 8.3
  $I(1+\gamma_1/\gamma, 0) = 0$ by 3).

==
Errata, P. 64, ll. -5 and -4

- $\boxed{(t,w) \to \delta(t,u)}$ should be $\boxed{(t,u) \to \delta(t,u)}$.
- $\boxed{\text{“... V of point } (x_0, y_0), \text{...”}}$ should be $\boxed{\text{“... V of the point } (x_0, y_0), \text{...”}}$.

==
On the use of the implicit function theorem, pp. 64-5

The implicit function theorem holds for a continuously differentiable mapping F of an open subset O of $\mathbf{R}^{n+k}$ into $\mathbf{R}^k$ at a point z_0 in O at which $F(z_0) = 0$, provided that the jacobian matrix $JF(z_0)$ of F at z_0 has maximal rank.[48] When this is so, we select a $k \times k$ submatrix of $JF(z_0)$ that is invertible and has column indices $j_1, \ldots, j_k$. Then the $j_1, \ldots, j_k$ components of the solutions z of the equation

$$F(z) = 0, \quad z \text{ in O},$$

which lie in an open neighborhood O_0 of z_0 can be expressed as continuously differentiable functions of the remaining n components. For example, if $\gamma_1'(t_0) \neq 0$ (respectively $\gamma_2'(t_0) \neq 0$) and F is the map $(t,u,x,y) \to (\gamma_1(t) - x, \gamma_2(t) + u - y)$ (respectively $(t,u,x,y) \to (\gamma_1(t) + u - x, \gamma_2(t) - y)$) of $(a,b) \times \mathbf{R}^3$ into $\mathbf{R}^2$. Consider $z_0 = (t_0, 0, x_0, y_0)$. Hence $F(z_0) = 0$,

$$JF(z_0) = \begin{bmatrix} \gamma_1'(t_0) & 0 & -1 & 0 \\ \gamma_2'(t_0) & 1 & 0 & -1 \end{bmatrix} \text{(respectively } JF(z_0) = \begin{bmatrix} \gamma_1'(t_0) & 1 & -1 & 0 \\ \gamma_2'(t_0) & 0 & 0 & -1 \end{bmatrix}\text{)},$$

and (i) holds since $F(t, 0, \delta(t,0)) = (0,0)$ if t is close enough to t_0.

==
Def., P. 65

For an illustration, see Figure 5.

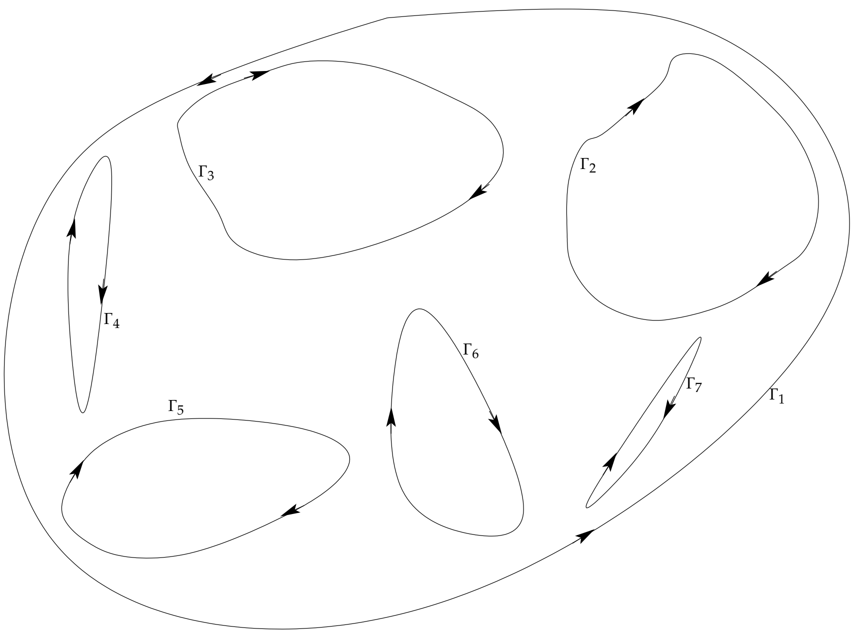

Figure 5: Oriented boundary.

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Errata, (1.1), P. 66

[48] A $k \times (n+k)$ matrix is said to have *maximal rank* if it has a $k \times k$ submatrix that is invertible.

$f(x_0 y_0)$ should be $f(x_0, y_0)$.

==

(2.1) $\Leftrightarrow$ (2.2), P. 67

Define $\alpha(u) = \left(\frac{f(z_0+u)-f(z_0)}{u} - c\right)\frac{u}{|u|}$ for $u \in \mathbf{C} - \{0\}$, whatever c you consider. Therefore, since $|\alpha(u)| = \left|\frac{f(z_0+u)-f(z_0)}{u} - c\right|$, it follows that

$$c = f'(z_0) \Leftrightarrow \lim_{u \to 0} \alpha(u) = 0.$$

==

2.2.3, PP. 68-9

- For "*Let f be a holomorphic function in a connected open set* D*; if the real part of f is constant, then f is constant.*", "*... real part ...*" can be replaced by "*... imaginary part ...*".[49] Additionally, for the necessity of the connectedness hypothesis, see (3.5) on P. 68, the third line on P. 69 and the sentence that follows (2.1) on P. 52. (This also replaces the conclusion (three last lines) of the proof.)

- For "...; g is holomorphic ...", use that the composition of holomorphic functions is holomorphic.

==

CAUCHY'S THEOREM, PP. 69-72

- *First Proof*
  To be more specific, "..., by the Green-Riemann formula (§1, formula (3.1)), ..." should be changed to "..., by PROP. 3.1, P. 55, ... ".

- *Second Proof*
  - **Errata**, P. 70, l. 10
    "Thus among <u>there</u> four ..." should be "Thus among <u>the</u> four ...".
  - "By the Cauchy criterion of convergence, there is a unique point z_0 common to all the rectangles $\mathrm{R}^{(k)}$.", P. 70, ll. -6 and -5
    As a matter of fact, a sequence of non-empty, compact, nested sets converges to its (non-empty) intersection.
  - "On the right hand side of (4.3), the first two integrals are zero and the third is negligible compared with the area of the rectangle $\mathrm{R}^{(k)}$ as k increases indefinitely; it is then negligible compared with $\frac{1}{4^k}$. Comparing this with (4.2) shows that we must have $\alpha(\mathrm{R}) = 0$; ...", final part of the proof
    * The first two integrals are zero since dz and $(z - z_0)dz$ are closed in D by the *First Proof*.
    * For the third integral, let $a\left(\mathrm{R}^{(k)}\right)$ be the area of $\mathrm{R}^{(k)}$ and $l\left(\mathrm{R}^{(k)}\right)$ its perimeter. Therefore[50] (see Figure 6)

$$\left|\int_{\gamma\left(\mathrm{R}^{(k)}\right)} \varepsilon(z)|z - z_0|\, dz\right| < a\left(\mathrm{R}^{(k)}\right) \cdot l\left(\mathrm{R}^{(k)}\right);$$

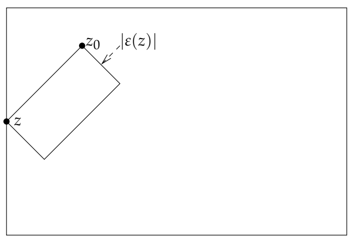

Figure 6: A rectangle of area $|\varepsilon(z)(z - z_0)|$ inside $\mathrm{R}^{(k)}$ with $z \in \gamma\left(\mathrm{R}^{(k)}\right)$.

[49] In fact, repeat the same argument, but now with $\frac{1}{2i}(f - \overline{f})$.

[50] $\left|\int_\gamma f\right| \leq (\max\{|f(z)| \mid z \in \gamma\}) \cdot (\text{length of } \gamma)$ if f is a continuous function on a smooth curve γ.

it is then $\ll a\left(\mathrm{R}^{(k)}\right)$ for k large enough. Now assume, without loss of generality, that the base and height of R are $\leq$ a unit length. Hence

$$a\left(\mathrm{R}^{(k)}\right) \overbrace{=}^{\text{P. 70, ll. -11, -10 and -9}} \frac{\text{area of R}}{4^k} \leq \frac{1}{4^k}.$$

Hence, for k large enough,

$$0 \leq \frac{1}{4^k}\,|\alpha(\mathrm{R})| \leq \left|\alpha\left(\mathrm{R}^{(k)}\right)\right| \ll \frac{1}{4^k}.$$

- *"Let $f(z)$ be an continuous ..."* should be *"Let $f(z)$ be a continuous ..."*.

==
Errata, THEO. 2, P. 72
"... with respect ot a ..." should be *"... with respect to a ..."*.
==
2.2.6, P. 73

- **Errata**, THEO. 3
  "Let $f(z)$ he ..." should be *"Let $f(z)$ be ..."*.

- Generalization of THEO. 3
  Let $f(z)$ be a holomorphic function in the open disc $|z - z_0| < \rho$; then f can be expanded as a power series in this disc, that is, there is an $\mathrm{S}(\mathrm{X}) \in \mathbf{C}[[\mathrm{X}]]$ with $\rho(\mathrm{S}) \geq \rho$ such that $f(z) = \mathrm{S}(z - z_0)$ for $|z - z_0| < \rho$.
  Proof. Consider $u = z - z_0$ and $g(u) = f(z)$. By THEO. 3, there is an $\mathrm{S}(\mathrm{X}) \in \mathbf{C}[[\mathrm{X}]]$ with $\rho(\mathrm{S}) \geq \rho$ such that $g(u) = \mathrm{S}(u)$ for $|u| < \rho$, that is, $f(z) = \mathrm{S}(z - z_0)$ for $|z - z_0| < \rho$.

- For "... the uniqueness of the power series expansion of a function in a neighbourhood of 0." at the very beginning of the *Proof*, see 1.2.8, P. 26.

- **Errata**, ll. -10 and -9
  "|ntegrate ... iz| $\leq r$:" should be "integrate ... $|z| \leq r$:".

==
==
III
==
==
3.1.1

- P. 79, ll. -11, -10 and -9
  - "We now propose to express the coefficients a_n in terms of integrals involving the function f."
    Hasn't it already been done? (See (6.1), P. 73.)

- P. 80, ll. 8-11
  - "... let $\mathrm{M}(r)$ be the upper bound of $|f(re^{i\theta})|$ as θ varies, ..."
    $\mathrm{M}(r)$ exists since *the image of a compact set under a continuous map is a compact set*.[51]
    (Here, the upper bound means the least upper bound!)
  - "The absolute value of the right hand side of (1.2) is then bounded above by $\mathrm{M}(r)$, ..."
    See the footnote that precedes the previous one.

==
3.2.2, P. 82

- *Proof*, THEO. 1

[51] Refer to the previously-mentioned Lima's Book, **Prop. 4**, p. *179*.

– "...; by multiplying f by a complex constant if necessary, we can reduce the theorem to the case when $f(a)$ is real and > 0, ..."
  Multiply f by $\overline{f(a)}$ if necessary.
– $\mathrm{M}(r)$ exists by the previous footnote.
– "... whence $f(a) \leq \mathrm{M}(r)$ ..."
  See the footnote that precedes the previous one.
– "It follows that the function
$$g(z) = \mathrm{Re}(f(a) - f(z))$$
  is ≥ 0 for sufficiently small $|z-a| = r$, and that $g(z) = 0$ if and only if $f(z) = f(a)$."
  In fact,
$$\mathrm{Re}(f(a) - f(z)) = \mathrm{Re}(f(a)) - \mathrm{Re}(f(z)) = |f(a)| - \mathrm{Re}(f(z))$$
  since $f(a)$ is real and positive. Hence, for any z sufficiently near to a, $g(z) \geq 0$ since
$$|f(a)| \geq |f(z)| \overbrace{\qquad\geq\qquad}^{\mathrm{Re}(f(z)) = |f(z)|\cos\theta} |\mathrm{Re}(f(z))| \geq \mathrm{Re}(f(z)).$$
  Now suppose $g(z_0) = 0$ and $f(z_0) \neq f(a)$ for z_0 close enough to a. Therefore
$$|f(a)| = \mathrm{Re}(f(z_0)) \text{ and } \mathrm{Im}(f(z_0)) \neq 0$$
  implies that
$$|f(z_0)| = \sqrt{|f(a)|^2 + \mathrm{Im}(f(z_0))^2} > |f(a)|,$$
  which is absurd.
– "By (2.1), the mean value of $g(z)$ on the circle
$$|z-a| = r$$
  is zero; ..."
  Since $\mathrm{Re}(-)$ is continuous,
$$\frac{1}{2\pi}\int_0^{2\pi}\left(f(a) - \mathrm{Re}\left(f\left(a + re^{i\theta}\right)\right)\right)d\theta = \frac{1}{2\pi}(f(a)2\pi) - \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right)d\theta\right).$$

• COR. and its *Proof*

– Here, *the upper bound* means *the <u>least</u> upper bound*. (Idem for $\boxed{\text{"Let M}' \text{ be the upper bound ..."}}$.)
– **Errata:** Replace $\boxed{\text{"... at at least ..."}}$ by $\boxed{\text{"... at least ..."}}$.
– "...; theorem 1 also shows that the subset of D where f takes the value $f(a)$ is open, and, as it is obviously closed ..."
  Consider $\mathrm{D}_a = \{z \in \mathrm{D} \,|\, f(z) = f(a)\}$, which is a nonempty set since $a \in \mathrm{D}_a$. Then:
  ∗ (Closedness part) Let $z_0 \in \overline{\mathrm{D}_a}$. Hence $z_0 = \lim_{n\to\infty} z_n$ with each $z_n \in \mathrm{D}_a$. Then, since f is continuous, $f(z_0) = \lim_{n\to\infty} f(z_n) = f(a)$, which implies that $z_0 \in \mathrm{D}_a$. Therefore $\overline{\mathrm{D}_a} \subset \mathrm{D}_a$, that is, D_a is closed;[52]
  ∗ (Openness part) Suppose that D_a is not open. Hence there is a $z_0 \in \mathrm{D}_a - \{a\}$ such that each open disk centered at z_0 intersects $\mathrm{D} - \mathrm{D}_a$. Thus, for each positive integer n, there is a $z_n \in \mathrm{D} - \mathrm{D}_a$ such that $z_0 = \lim_{n\to\infty} z_n$. Then, since f is continuous, $f(z_0) = \lim_{n\to\infty} f(z_n) \neq f(a)$, which is absurd since $z_0 \in \mathrm{D}_a$.

==

Schwarz' Lemma, P. 83

• THEOREM

– The hypothesis can be rephrased as

[52]Refer to the previously-mentioned Lima's Book, **Cor. 1**, p. *81*.

$\boxed{\textit{Let } f(z) \textit{ be a holomorphic function from the open unit disk to itself such that } f(0) = 0.}$.

– The second conclusion can be rephrased as

$$\boxed{f(z) \textit{ is a rotation if } |f(z_0)| = |z_0| \textit{ for some } z_0 \neq 0.}$$

- *Proof*
  - $\boxed{\text{“It follows that } f(z)/z \text{ is holomorphic for } |z| < 1.\text{”}}$
    Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. Hence $g(z)$ is continuous in $|z| < 1$.[53] Now apply COR., THEO. 4, P. 74.
  - $\boxed{\text{“This inequality holds also for } |z| \leq r \text{ because of the maximum modulus principle.”}}$
    See *Note*, P. 83.

==
LAURENT'S SERIES, P. 84

- $g(u)$, which is associated with $\sum_{n<0} a_n X^{-n}$, comes from

$$\sum_{n<0} a_n u^{-n} \overbrace{=}^{m=-n} \sum_{m>0} a_{-m} u^m.$$

- $f_2'(z)$ comes from

$$f_2(z) = f_2(u^{-1}) = \sum_{n<0} a_n u^{-n} \overbrace{=}^{m=-n} g(u)$$

- hence $f_2'(z) = g'(u)$, with $' = \frac{d}{dz}$ - and

$$-\frac{1}{z^2}\sum_{n>0} n a_{-n} z^{1-n} = \sum_{n>0} -n a_{-n} z^{-n-1} \overbrace{=}^{m=-n-1} \sum_{m<-1} (m+1) a_{m+1} z^m \overbrace{=}^{m=p-1} \sum_{p<0} p a_p z^{p-1}.$$

- “The convergence of series (1.3) is normal in any annulus $r_2 \leq |z| \leq r_1$, with

$$\rho_2 < r_2 < r_1 < \rho_1.\text{”}$$

  See PROP. 3.1, P. 19.

==
“This extension is obviously unique if it exists (by the principle of analytic continuation, or, in this case, simply because of continuity).”, P. 87, ll. -6, -5 and -4
Suppose that f_1 and f_2 are holomorphic extensions of f to $\mathrm{D} = \{z : |z| < \rho\}$. Therefore:

- (Principle of analytic continuation)[54]
  Let N be a sufficiently small neighbourhood of a point of D such that $0 \notin \mathrm{N}$. Then, since $f_1(z) = f(z) = f_2(z)$ for each $z \in \mathrm{N}$, it follows that $f_1(z) = f_2(z)$ for all $z \in \mathrm{D}$.
- (Continuity)
  It suffices to prove that $f_1(0) = f_2(0)$. In fact, since $f_1(z)$ and $f_2(z)$ are continuous, $\lim_{z\to 0} f(z) = \lim_{z\to 0} f_1(z) = f_1(0)$ and $\lim_{z\to 0} f(z) = \lim_{z\to 0} f_2(z) = f_2(0)$. Now use the uniqueness of the limit.

==
1 *st. case*, P. 88
For the sake of consistency,[55] $g(0) \neq 0$ must hold. In fact, $f(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots$ with $a_{-n} \neq 0$, that is, $g(0) = a_{-n} \neq 0$.
==
THEO. (Weierstrass) and its *Proof*, PP. 88-9

[53] $\lim_{z\to 0} g(z) = g(0)$ by applying L'Hopital's Rule.
[54] See P. 40.
[55] Refer to P. 42.

- Actually, Casorati-Weierstrass Theorem.
- Obviously, the THEO. holds for each $\varepsilon \in (0, \rho]$.
- A reasoning to be considered right before the last sentence
  If $g(0) \neq 0$, since $g(z)(f(z)-a) = 1$, then $f(z)$ is bounded in some neighbourhood of 0, that is, $f(z)$ can be extended to a holomorphic function on the entire disc $|z| < \rho$,[56] which is absurd.[57] Then $g(0) = 0$ is the only possibility.

==

(1. 1), P. 89
Consider $\mathrm{P}_t(\mathrm{M}) = \mathrm{P} + t(\mathrm{M} - \mathrm{P}) = (tx, ty, 1 + t(u-1))$ for each $t \geq 0$. Hence

$$1 + t_0(u-1) = 0 \Rightarrow t_0 = \frac{1}{1-u} \Rightarrow \mathrm{P}_{t_0}(\mathrm{M}) = \left(\frac{x}{1-u}, \frac{y}{1-u}, 0\right).$$

==

$\boxed{\text{"... the residue of } f \text{ at infinity is } -a_{-1}.\text{"}}$, P. 92, l. 7
In fact, the residue of $-\frac{1}{z'^2} f\left(\frac{1}{z'}\right)$ at $z' = 0$ is

$$-z'^{-2} \sum_n a_n z'^{-n} = -\sum_n a_n z'^{-(n+2)} = \cdots - a_1 z'^{-3} - a_0 z'^{-2} - a_{-1} z'^{-1} - a_{-2} - a_{-3} z' - a_{-4} z'^2 - \cdots.$$

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(3.2), P. 94
Use l'Hôpital's rule.

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"... gives

$$f'/f = \frac{k}{z - z_0} + g'/g;$$

thus f'/f has z_0 as a *simple pole* and *the residue of this pole is equal to the integer k, ...*", P. 96, ll. 2-4

In fact, by the first *Definition* on P. 88, z_0 is not an isolated singularity of g'/g. Now use the necessary and sufficient condition that follows immediately after that *Definition*.

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[56] Refer to PROP 4.1, P. 87.
[57] See the first *Def.*, P. 88.