Two new augmented Lagrangian algorithms with quadratic penalty for equality problems

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Abstract In this paper we present two augmented Lagrangian methods applied to nonlinear programming problems with equality constraints. Both use quadratic penalties and the structure of modern methods to problems with inequality constraints. Therefore, they can be seen as augmented Lagrangian applied to problem with inequality constraints extended to problems with equality constraints without additional of slack variables. In the main result of the paper, we show that under conventional hypotheses the augmented Lagrangian function generated by the two methods has local minimizer, as in the case of the proposed method by Hestenes [11] and Powell [22]. Comparative numerical experiments on CUTEr problems are presented to illustrate the performance of the algorithms.

Keywords Nonlinear programming \cdot augmented Lagrangian methods \cdot penalty function \cdot numerical experiments

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1 Introduction

In the classical augmented Lagrangian methods the quadratic penalty introduced by Hestenes [11] and Powell [22] is applied to problems with equality constrains

minimize
$$f(x)$$

subject to $h(x) = 0$ (1)
 $x \in \mathbb{R}^n$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions.

The augmented Lagrangian function propose for these authors is defined by

$$\mathcal{L}(x,\lambda,\rho) = f(x) + \sum_{i=1}^{m} \left\{ \frac{\rho}{2} \left[h_i(x) \right]^2 + \lambda_i h_i(x) \right\}$$
(2)

where, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ is the vector of Lagrange multipliers and $\rho > 0$ is the penalty parameter, in which, for $\rho > \bar{\rho} > 0$ the augmented Lagrangian method converge to a solution of the problem (1).

In this paper, we propose two new augmented Lagrangian methods to solve the problem (1). The methods which we are proposing are extentions of the results showed in [17] and [27] for nonlinear programming problems with inequality constrains, given by

minimize
$$f(x)$$

subject to $g(x) \le 0$ (3)
 $x \in \mathbb{R}^n$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions.

Almost all proposed augmented Lagrangian methods to solve the problem (3), use the following methodology: define the dual problem associated to problem (3), that is in the primal format, and the proximal point algorithm associated with the dual problem. Due to the structure of the dual problem to be simpler than the primal, from the theoretical point of view, the convergence is easier to be treated in the dual. On the other hand, the computation is simpler to be treated in the primal problem. Therefore, using some distance measure, for example, Euclidian norm, ϕ -divergences, Bregman distance, it is possible to develop methods applied to the dual problem, with good properties of convergence and that under some hypothesis are equivalents with the augmented Lagrangian method in the primal.

Rockfellar, in [23,24], was one of the first to use this methodology through proximal point algorithm for maximal monotone operator.

Many papers were published about the augmented Lagrangian methods applied to problem (3), for example, [1,4–6,8,12–17,19,20,25–27]. Initially, we are concerning in the papers [17] and [27] because we are also extending these to problems in the form of (1) without introducing slack variables.

Next, we relate some important facts about these modern augmented Lagrangian methods to the problems with inequality constrains (3), that also motivate us to generalize related to the problems with equality constrains (1):

- The convergence of the method is guaranteed even if the penalty parameter is kept constant, however, if the penalty parameter is decreased, the convergence will be faster [17,27];
- The existence of a certain equivalence relation between trust region and the dualkernel proposed, in such a way that by increasing the penalty parameter, trust region is decreased [17];
- For the quadratic particular case and depending on the choice of penalty parameter, the behavior of the proximal point algorithm (here looking at the dual) is similar to the affine scaling algorithm, which is known to have good convergence properties [17];
- Numerical tests with several problems has shown that these methods perform well, even for large problems including bounds on the variables. Birgin, Castillo and Martinez in [3], tested 65 implementations of different augmented Lagrangian methods. For the class of problems tested, the authors concluded that the classical methods show superiority to the modern ones, enhancing our motivation to introduce new augmented Lagrangian methods similar to the classics, but with the properties of the modern ones.

The remaining sections of this paper are organized as follows: in Section 2, we describe the augmented Lagrangian functions with the new penalty functions associated with problem (1). In Section 3, we show the existence of a strict local minimizer of the augmented Lagrangian functions, as well as, the two algorithms of the proposed methods. The first is related to the new functions proposed and the second, for comparison purposes, the classical method of Hestenes [11] and Powell [22].

Numerical tests on CUTEr problems, suggested in [7], were performed to compare the performance of the presented methods and results are described and analyzed in Section 4. Conclusions and perspectives are presented in Section 5.

2 Introduction of new penalties

Associated with the problem (1) we define the augmented Lagrangian functions

$$(x,\lambda,\gamma) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{++} \to \mathcal{L}(x,\lambda,\gamma) = f(x) + \sum_{i=1}^m \left\{ \frac{\gamma_i}{2} \left[h_i(x) \right]^2 + \lambda_i h_i(x) \right\}$$
(4)

$$(x,\lambda,\beta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{++} \to \mathcal{L}(x,\lambda,\beta) = f(x) + \sum_{i=1}^m \left\{ \frac{\beta_i}{2} \left[h_i(x) \right]^2 + \lambda_i h_i(x) \right\}$$
(5)

where, λ is the vector of Lagrange multipliers associated with h(x) = 0 and γ and β are the penalties parameters. The penalty parameter γ_i in (4) is defined as

$$\gamma_i = \frac{\lambda_i^2}{r_i}, \text{ with } r_i > 0$$
 (6)

and β_i in (5) is defined as

$$\beta_i = \frac{\lambda_i}{r_i}, \text{ with } r_i \neq 0,$$
 (7)

for all i = 1, ..., m. As the penalties parameters must be positive, we will choose r_i such that this occur, for example, in (7) $r_i < 0$ if $\lambda_i < 0$ and $r_i > 0$ otherwise.

Note the presence of γ_i in (4) and of β_i in (5) in the term that penalizes the constraints, both are dependents of the multipliers. This is the difference in relation to the classical penalty (2) and this difference and its importance will be detailed later.

Consider the quadratic function of one real variable

$$y \in \mathbb{R} \to \theta(y) = y^2 \tag{8}$$

and the Lagrangian function associated with (1)

$$(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m \to \ell(x,\lambda) = f(x) + \sum_{i=1}^m \lambda_i h_i(x).$$
 (9)

Thus, (2), (4) and (5) can be rewritten, respectively, as follows:

$$\mathcal{L}(x,\lambda,\rho) = \ell(x,\lambda) + \frac{\rho}{2} \sum_{i=1}^{m} \theta(h_i(x)),$$
(10)

$$\mathcal{L}(x,\lambda,\gamma) = \ell(x,\lambda) + \frac{1}{2} \sum_{i=1}^{m} \theta(\sqrt{\gamma_i}h_i(x))$$
(11)

and

$$\mathcal{L}(x,\lambda,\beta) = \ell(x,\lambda) + \frac{1}{2} \sum_{i=1}^{m} \beta_i \theta(h_i(x)).$$
(12)

where, ℓ is the function (9) and θ is the function (8).

In [17] and [27] with methodologies similar to (11) and (12), respectively, the authors present a family \mathcal{P} of penalty functions varying the θ function. Both construct augmented Lagrangian based on problems with inequality constrains. The augmented Lagrangian (11) with θ given by barrier function of Polyak in [21] is used in [17] to present augmented Lagrangian methods equivalent to proximal point with Bregman distances in the dual. The augmented Lagrangian function (12) with θ being the exponential function is used in [27] to present augmented Lagrangian methods equivalent to proximal point with ϕ -divergences.

In the augmented Lagrangian methods the penalty parameter is one of few that we have freedom of choice for its update. In Lemma 1, we show that for a particular choice of the penalty parameter, the three augmented Lagrangian functions are equivalents. This fact will be important in the update penalty parameter in the augmented Lagrangian algorithms that will be presented in this paper.

Lemma 1 If $\lambda_i \neq 0$, for all i = 1, ..., m, and the penalties parameters γ in (11) and β in (12) are updated as $\gamma_i = \frac{\lambda_i^2}{r_i}$ (given by 6) with

$$r_i = \frac{\lambda_i^2}{\rho} \tag{13}$$

and
$$\beta_i = \frac{\lambda_i}{r_i}$$
 (given by 7) with
 $r_i = \frac{\lambda_i}{\rho}$
(14)

and $\rho > 0$ (given by 10), then the functions (10), (11) and (12) are the same.

Proof: The proof is immediate. We will make for one case, the other is identical.

In fact replacing $\gamma_i = \frac{\lambda_i^2}{r_i}$ with $r_i = \frac{\lambda_i^2}{\rho}$ in (11) and using θ given by (8), follow that (10) and (11) are the same.

Figure 1 shows geometrically, in one bidimensional case with r = 1, the behavior of the classical penalty, $\theta(y) = y^2$, and of new penalties, $\theta(\sqrt{\gamma}y) = \lambda^2 y^2$ and $\beta\theta(y) = \lambda^2 y^2$ λy^2 , being θ a real and quadratic function defined in (8). In the first three graphics, we consider $\lambda = 2$ and the following three, $\lambda = 1/2$. Note the influence of λ in the range of graphics, we see that $\theta(\sqrt{\gamma}y)$ has the characteristic of penalizing more than the other two in case $\lambda > 1$ and less if $\lambda < 1$. On the other hand, $\beta \theta(y)$ is an intermediate to the other two. Obviously, if $\gamma = 1$ and $\beta = 1$ the three coincide. This fact is reinforced in [17] when the problem is linear with inequality constraints, which leads, in the dual, the proximal point algorithms with quadratics kernel that are equivalent to the affine scaling method. These are some points we consider relevant compared with the classical penalty. Intuitively and observing the geometric aspects of proposed penalties, we believe that there are some updates of penalty parameter in terms of λ , that increases the speed of convergence. This fact was used, although intuitively, in the methods we implemented. We will continue investigating these properties, trying to set theoretical results that ensure our statements.

Fig. 1: Penalties $\theta(y)$, $\theta(\sqrt{\gamma}y)$ and $\beta\theta(y)$ for two distinct values of λ

3 Augmented Lagrangian with the new penalty functions

In this section we analyze the properties that are satisfied by the functions (4) or (11) and (5) or (12). We show that, just a finite value of γ and β so that \bar{x} , a point that satisfies the second order sufficient condition for problem (1), is a strict local minimizer of the augmented Lagrangian functions (4) and (5).

Firstly, consider Ω the feasible set for problem (1), in other words

$$\Omega = \{ x \in \mathbb{R}^n : h(x) = 0 \}.$$

$$(15)$$

In the following we present a Lemma that is known in the literature and will be used in the proof of main theorem of this paper.

Lemma 2 Consider $G = G^T \in \mathbb{R}^{n \times n}$ such that $z^T G z > 0$ for all $z \in \mathcal{N}(A)$, $z \neq 0$ and $A \in \mathbb{R}^{m \times n}$. Then, exist $\bar{\rho} \ge 0$ such that $G + \rho A^T A > 0$ for all $\rho \ge \bar{\rho}$.

If ρ is considered vetorial, $\rho = (\rho_1, \rho_2, \dots, \rho_n)$, the Lemma 2 still valid, just write $G + diag(\rho)A^T A$ such that $\rho_i \ge \overline{\rho}$ for all $i = 1, \dots, n$.

For the next theorem we use the assumption that the problem (1) satisfies the Linear independence constraint qualification (LICQ).

Definition 1 A point $\bar{x} \in \Omega$ is said to be a regular point if the gradients of the constraints of the problem (1) are linearly independent. We call it the qualification condition LICQ.

Theorem 1 (Second order sufficient conditions)

Suppose that for any feasible point $\bar{x} \in \mathbb{R}^n$, there is a vector of Lagrange multipliers $\bar{\lambda}$ such that the KKT conditions are satisfied for the problem (1). Also suppose that, $z^T \nabla_{xx}^2 \ell(\bar{x}, \bar{\lambda}) z > 0$, for all $z \in \mathcal{N}(\nabla h(\bar{x}))$, $z \neq 0$. Then \bar{x} is a strict local minimizer for the problem (1).

Proof : See [18].

The convergence of the classical augmented Lagrangian method is discussed by Bertsekas [2]. In this paper, the author shows that under the hypotheses of the theorem 1 and for any given bounded set $\mathbb{Y} \subset \mathbb{R}^m$ there exists a scalar $\rho^* \ge 0$ such that for all $\rho > \rho^*$ and for all $\lambda \in \mathbb{Y}$ the function $\mathcal{L}(x, \lambda, \rho)$ has a unique minimization point $x(\lambda, \rho)$ within some open ball centered at \overline{x} . Furthermore, for some scalar M > 0 follow that

$$\|x(\lambda,\rho) - \overline{x}\| \le \frac{M \|\lambda - \overline{\lambda}\|}{\rho}, \quad \forall \rho > \overline{\rho}, \quad \lambda \in \mathbb{Y}$$
(16)

and

$$\left\|\overline{\lambda}(\lambda,\rho) - \overline{\lambda}\right\| \le \frac{M \left\|\lambda - \overline{\lambda}\right\|}{\rho}, \quad \forall \rho > \overline{\rho}, \quad \lambda \in \mathbb{Y}.$$
(17)

After presenting the previous results, we are ready to show the main contribution of this paper.

Theorem 2 Suppose that \bar{x} is a local solution of the problem (1) in which the qualifying condition LICQ and the conditions of the theorem 1 are satisfied. Then, there is a limit value $\bar{\gamma} \ge 0$ such that, for all $\gamma > \bar{\gamma}$, \bar{x} is a minimizer of augmented Lagrangian function $\mathcal{L}(x, \bar{\lambda}, \gamma)$ given by (4).

Proof: We will prove the result showing that $\bar{x} \in \Omega$ satisfies the second order sufficient conditions for $\mathcal{L}(x, \bar{\lambda}, \gamma)$, that is

$$\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = 0 \text{ and } \nabla^2_{xx} \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) > 0.$$

Let \mathcal{L} the augmented Lagrangian function (4), it follows that

$$\nabla_{x} \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = \nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_{i} \nabla h_{i}(\bar{x}) + \sum_{i=1}^{m} \gamma_{i} h_{i}(\bar{x}) \nabla h_{i}(\bar{x})$$

or
$$\nabla_{x} \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = \nabla_{x} \ell(\bar{x}, \bar{\lambda}) + \sum_{i=1}^{m} \gamma_{i} h_{i}(\bar{x}) \nabla h_{i}(\bar{x})$$

By hypothesis, the KKT conditions in $\bar{x} \in \Omega$ are satisfied, then $\nabla_x \ell(\bar{x}, \bar{\lambda}) = 0$ and $h(\bar{x}) = 0$.

Therefore, $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = 0$. Thus, \bar{x} is also a stationary point of the function (4). Now,

$$\nabla_{xx}^2 \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = \nabla^2 f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \nabla^2 h_i(\bar{x}) + \sum_{i=1}^m \gamma_i \nabla h_i(\bar{x})^T \nabla h_i(\bar{x}) + \sum_{i=1}^m \gamma_i h_i(\bar{x}) \nabla^2 h_i(\bar{x})$$

or
$$m$$

 $\nabla_{xx}^2 \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = \nabla_{xx}^2 \ell(\bar{x}, \bar{\lambda}) + \sum_{i=1}^m \gamma_i \nabla h_i(\bar{x})^T \nabla h_i(\bar{x}) + \sum_{i=1}^m \gamma_i h_i(\bar{x}) \nabla^2 h_i(\bar{x})$ as $\bar{x} \in \Omega$, it follows that $h(\bar{x}) = 0$ and

$$\nabla^2_{xx} \mathcal{L}(\bar{x}, \bar{\lambda}, \gamma) = \nabla^2_{xx} \ell(\bar{x}, \bar{\lambda}) + (diag(\gamma) \nabla h(\bar{x}))^T \nabla h(\bar{x}).$$

By hypothesis, $\nabla_{xx}^2 \ell(\bar{x}, \bar{\lambda}) > 0$. Therefore, by Lemma 1 the proof is complete. \Box

Note 1 The theorem 2 is also valid for the augmented Lagrangian function (5). However, an adaptation is necessary in this demonstration in order to Hessian of \mathcal{L} is positive definite. In fact, let \mathcal{L} be the function (5). Repeating the steps of the proof of the Theorem 2, we obtain the following expression for the Hessian of \mathcal{L}

$$\nabla^2_{xx} \mathcal{L}(\bar{x}, \bar{\lambda}, \beta) = \nabla^2_{xx} \ell(\bar{x}, \bar{\lambda}) + (diag(\beta) \nabla h(\bar{x}))^T \nabla h(\bar{x}).$$
(18)

Since $\beta_i = \frac{\lambda_i}{r_i}$, we take $r_i > 0$ if $\lambda_i < 0$ and $r_i < 0$ otherwise. This was the strategy used in the augmented Lagrangian algorithm for \mathcal{L} given by the function (5).

The convergence of the augmented Lagrangian methods with the function (11) and (12) is set analogously and uses the same idea presented by Bertsekas [2]. In this case consider the function (11), for any given bounded set $\mathbb{Y} \subset \mathbb{R}^m$ exists a scalar $\overline{\gamma} \geq 0$ such that for all $\gamma > \overline{\gamma}$, where $\gamma = \max{\{\gamma_i, i = 1, ..., m\}}$, and for all $\lambda \in \mathbb{Y}$ exists some open ball centered at \overline{x} and a scalar M > 0 such that

$$\|x(\lambda,\gamma) - \overline{x}\| \le \frac{M \|\lambda - \overline{\lambda}\|}{\gamma}, \quad \forall \gamma > \overline{\gamma}, \quad \lambda \in \mathbb{Y}$$
(19)

and

$$\left\|\overline{\lambda}(\lambda,\gamma) - \overline{\lambda}\right\| \le \frac{M \left\|\lambda - \overline{\lambda}\right\|}{\gamma}, \quad \forall \gamma > \overline{\gamma}, \quad \lambda \in \mathbb{Y}.$$
(20)

Below we present the augmented Lagrangian algorithm with the penalty functions proposed in this paper.

Algorithm 1

Data: $x^0 \in \mathbb{R}^n$, $\lambda^0 \in \mathbb{R}^m$, $\gamma^0 \in \mathbb{R}^{m_{++}}$, $\beta^0 \in \mathbb{R}^{m_{++}}_{++}$ k = 0While the stop criterion is not satisfied $x^{k+1} \in argmin \left\{ \mathcal{L}(x, \lambda^k, \gamma^k) \text{ or } \mathcal{L}(x, \lambda^k, \beta^k) : x \in \mathbb{R}^n \right\}$ UPDATE the Lagrange multipliers $\lambda_i^{k+1} = \lambda_i^k + \gamma_i^k h_i(x^{k+1}), \quad i = 1, ..., m$ (for the function 4) OR $\lambda_i^{k+1} = \lambda_i^k + \beta_i^k h_i(x^{k+1}), \quad i = 1, ..., m$ (for the function 5) UPDATE the penalty parameter, componentwise $\gamma_i^{k+1}, \quad i = 1, ..., m$ (eq. 6 for the function 4) OR $\beta_i^{k+1}, \quad i = 1, ..., m$ (eq. 7 for the function 5) k = k + 1

End

In Algorithm 1 the function \mathcal{L} refers to the augmented Lagrangian function (4) or (5). Recalling that, in the case of the function (5) the penalty parameter has to be updated as the previous Note (1). Other details about the choice and parameters update will be given in the description of numerical tests (Section 4).

To compare the efficiency of the proposed method, we also implemented the classical method of Hestenes [11] and Powell [22].

Algorithm 2

Data: $x^0 \in \mathbb{R}^n$, $\delta \in (0, 1)$, $\lambda^0 \in \mathbb{R}^m$, $\rho^0 \in \mathbb{R}_{++}$ k = 0

WHILE the stop criterion is not satisfied

$$\begin{aligned} x^{k+1} &\in argmin\left\{f(x) + \sum_{i=1}^{m} \left[\frac{\rho}{2} \left(h_i(x)\right)^2 + \lambda_i h_i(x)\right]\right\}\\ \lambda_i^{k+1} &= \lambda_i^k + \rho^k h_i(x^{k+1}), \quad i = 1, \dots, m\\ \text{IF} \left\|h(x^{k+1})\right\| &\geq \delta \left\|h(x^k)\right\|\\ &\text{DO} \ \rho^{k+1} &\geq \rho^k \end{aligned}$$

ELSE
$$\begin{array}{c} \text{DO} \ \rho^{k+1} &= \rho^k\\ \text{END}\\ k &= k+1 \end{aligned}$$

End

In the classical Algorithm 2, Hestenes [11] and Powell [22] updated the Lagrange multiplier, λ , forcing to satisfy the KKT conditions. In iteration *k*, are known λ^k , ρ^k and the algorithm determines x^{k+1} . If $\lambda_i^{k+1} = \lambda_i^k + \rho^k h_i(x^{k+1})$, then

$$\nabla_x \mathcal{L}(x^{k+1}, \lambda^{k+1}, \rho^k) = \nabla_x \mathcal{L}(x^{k+1}, \lambda^{k+1}) = 0$$
(21)

supposing that x^{k+1} is the solution of the problem (1).

Let us use this to justify the choice of λ^{k+1} in Algorithm 1. We will present only the case of the function (4), because the process is analogous to the function (5).

In fact, taking the derivative of (4) with respect to x and keeping λ^k and γ^k fixed, we obtain

$$\nabla_{x} \mathcal{L}(x, \lambda^{k}, \gamma^{k}) = \nabla f(x) + \sum_{i=1}^{m} \left(\lambda_{i}^{k} + \gamma_{i}^{k} h_{i}(x) \right) \nabla h_{i}(x)$$

And the gradient evaluated x^{k+1} results in

$$\nabla_x \mathcal{L}(x^{k+1}, \lambda^k, \gamma^k) = \nabla f(x^{k+1}) + \sum_{i=1}^m \left(\lambda_i^k + \gamma_i^k h_i(x^{k+1})\right) \nabla h_i(x^{k+1})$$

Therefore, if $\lambda_i^{k+1} = (\lambda_i^k + \gamma_i h_i(x^{k+1}))$ and x^{k+1} is the solution of the problem (1), then $\nabla_x \mathcal{L}(x^{k+1}, \lambda^{k+1}, \gamma^k) = \nabla_x \mathcal{L}(x^{k+1}, \lambda^{k+1}) = 0$.

4 Numerical tests

In this section we present the numerical tests performed to evaluate the performance of the proposed methods. The Algorithms 1 and 2 were implemented in Matlab 7.8 version R2009a, processor Intel(R) Core(TM)2 Duo CPU T5870 2GHZ and 3GB of memory RAM.

We implemented two variants of the Algorithm 1. Related to the first function, it was used the augmented Lagrangian (4) that we call the method LAPM, and the second one is augmented Lagrangian (5), that we call the method LAPB. We also implemented the Algorithm 2 with the function (2), that we call LAPC. Let us now discuss the main features of our implementation.

The update penalty parameter is based on a measure of infeasibility of the problem, since, there is no need to penalize in every iteration. Thus, if the measure of infeasibility, $||h(x^{k+1})|| \ge \delta ||h(x^k)||$, is not satisfied, it is because there was not a significant gain in viability and then the penalty parameter is increased. Therefore, in Algorithm 2 we consider an increase in the value of the penalty parameter doing $\rho^{k+1} = 2\rho^k$. For Algorithm 1, we describe below how were the penalty parameter γ^k and β^k updated, considering firstly the function (4) and then the function (5).

Consider in Algorithm 1 the function (4). As we proved in Lemma 1, if we define the penalty parameter (6) with $r_i = \frac{\lambda_i^2}{\rho}$, i = 1, ..., m, the proposed method becomes the classical method of Hestenes [11] and Powell [22].

Therefore, we chose to update *r* taking a multiple of $\frac{\lambda_i^2}{\rho}$, for all i = 1, ..., m, as follows (remember that the penalty parameter is γ and not *r*)

$$r_i^{k+1} = \alpha \frac{(\lambda_i^{k+1})^2}{2\rho^{k+1}} + (1-\alpha) \frac{(\lambda_i^{k+1})^2}{\rho^{k+1}} = \left(1-\frac{\alpha}{2}\right) \frac{\left(\lambda_i^{k+1}\right)^2}{\rho^{k+1}}$$
(22)

where $i = 1, ..., m, 0 \le \alpha \le 1$ and k = 0, 1, 2, ...

Concerning the case of the function (5) the penalty parameter (7) was updated considering (see Note (1)):

$$r_i^{k+1} = \alpha \frac{\lambda_i^{k+1}}{2\rho^{k+1}} + (1-\alpha) \frac{\lambda_i^{k+1}}{\rho^{k+1}} = \left(1 - \frac{\alpha}{2}\right) \frac{\lambda_i^{k+1}}{\rho^{k+1}}.$$
(23)

After some numerical experiments varying the value of α and maintaining $\rho^{k+1} = 2\rho^k$ (it was updated in the same way as in algorithm 2), we conclude that $\alpha = 0.85$ is the value that yielded better results for most problems tested.

For solving the unconstrained subproblems we use the CG-DESCENT algorithm, that establishes a new nonlinear Conjugate Gradient method based on an inexact line search, developed by Hager and Zhang [9,10]. This algorithm use only first order

information and the authors made comparisons with others methods and concluded that CG-DESCENT showed competitive.

We did tests with 134 problems from CUTEr collection in the form of the problem (1), with equality constraints and free variables. We consider problems of various dimensions, that is, problems with at least 2 variables and 1 constraint and at most 1984 variables and 1024 constraints. In the implementation we consider the following values: $\lambda^0 = (1, ..., 1)^T$, $\delta = 0.1$, $\rho^0 = 1$ and $\alpha = 0.85$. In all experiments we used the start points defined in the CUTEr collection.

The problems tested are listed in Table 1 in alphabetical order. For each problem, the results of the first row correspond to the LAPC method, the second to the LAPM and of the third one to the LAPB. We use n, m to represent the number of variables and the number of equality constraints, respectively. For both methods, *it.ext* is the number of iterations of the outer algorithm, *it.int* is the number of iterations of the outer algorithm, *it.int* is the number of iterations of the augmented Lagrangian function, $\#\nabla \mathcal{L}$ is the number of gradient evaluations of the augmented Lagrangian function, $f(\bar{x})$ is final value of the objective function, $\|h(\bar{x})\|$ it is the Euclidean norm of the constraint solution and *exit* represents the stopping criterion.

The stopping criterion was based on the values of $c_1 = ||\nabla f(x^k) + \nabla h(x^k)^T \lambda||$, $c_2 = ||h(x^k)||$, $c_3 = it.ext$, $c_4 = it.int$, $c_5 = search$ and $c_6 = time$ in seconds. Thus, we interrupt the execution of Algorithms 1 and 2, for some k, when any of the following criteria was satisfied:

IF $c_1 \le 10^{-3}$ and $c_2 \le 10^{-5}$ exit=1 ELSEIF $c_1 \le 10^{-2}$ and $c_2 \le 10^{-8}$ exit=2ELSEIF $c_1 \le 10^{-8}$ and $c_2 \le 10^{-2}$ exit=3ELSEIF $c_1 \le 10^{-10}$ and $c_2 \le 10^{-1}$ exit=4ELSEIF $c_1 \le 10^{-1}$ and $c_2 \le 10^{-10}$ exit=5 Elseif $c_3 \ge 500$ or $c_4 > 10000$ exit=6 Elseif $c_5 \ge 1000$ exit=7Elseif $c_6 > 3600$ exit=8End

Values 1 to 8 were assigned to *exit*, indicating the stopping criterion satisfied. For example, for the first problem shown in the table 1, ARGTRIG, both Algorithm 1, with functions (4) and (5), and Algorithm 2, were interruped when the algorithm exceeded the maximum number of searches established.

It should be noted that the stopping criteria represented by the values from 1 to 5 *exit*, take into account the same conditions, feasibility (c_2) and stationarity (c_1) ,

however, with small variations in the precision. We establish such variation in order to reduce the computational effort of the algorithms to satisfy a very strict stopping criterion, because we observed that for problems, such as, GENHS28 with n = 10 and m = 8, GOTTFR, HIMMELBA, HIMMELBC, HS6, HS8, HS39, among others, even when the algorithm had found the solution, the iterative process took up to be stopped.

o.m. o.m. <th< th=""><th>Problem</th><th>Method</th><th>it.ext</th><th>it.int</th><th>search</th><th>#L</th><th>#VL</th><th>$f(\bar{x})$</th><th>$h(\bar{x})$</th><th>exit</th><th>time</th></th<>	Problem	Method	it.ext	it.int	search	#L	#VL	$f(\bar{x})$	$ h(\bar{x}) $	exit	time
AKCTRIG LAPM 22 805 1006 8250 7580 0.0000 1.11678-09 7 881.40 C002.00) LAPE 22 6656 1006 12322 10734 0.0000 6.4112-00 7 881.43 C002.00) LAPE 22 67 1 62 33 -1.0006 6.059966 1 2.01 C.D.1 LAPE 24 1533 1 62 33 -1.0006 6.059966 1 2.01 C.J.1 LAPE 24 65 2 91 53 -1.0006 6.059966 1 2.01 C.J.1 LAPE 20 72 8 61 38 0.0326 9.07316-09 2 2.50 C.J.1 LAPE 20 5 164 87 2.165 2.63 2.50 C.J.1 LAPE 21 122 4.55 2.50 2.50 2.50 2.50 2.50 C.J.	(n, m)										(seconds)
CODE LAPM 22 73.6 1007 906 1928 0.0000 1.131-16.8 7 88.1.4 BROWNALE LAPR 23 1471 3 19538 10794 0.0000 5.4100-0 6.412-0 7 88.32 BROWNALE LAPR 23 1473 10 47331 3.131 0.0000 1.2711-00 7 2.91 COLD LAPR 23 455 2 91 33 -1.0006 6.0599-06 1 2.91 G17 LAPR 8 55 2 91 33 -1.0006 6.0599-06 1 2.91 G17 LAPR 19 102 5 164 87 2.1897 8.4382-00 2.9 3.61 G3.1 LAPR 23 182 290 16.3 4.053 5.0744-00 2.9 3.61 G3.5 LAPR 21 182 290 153 4.095 5.0744-00 2.9	ARGTRIG	LAPC	27	8605	1006	8236	7580	0.0000	1.1679e-09	7	1206.60
LAPB 22 6656 1006 12252 10234 0.0000 6.4412-09 7 88.38 (200,00) LAPM 26 4278 1004 45283 28113 0.0000 1.9216-07 291.90 BT LAPB 24 11534 17145 79708 0.0000 4.9372-69 2 250 BT LAPM 8 55 2 91 53 -1.0006 6.6997.60 1 2.101 HT LAPM 8 55 2 91 53 -1.0006 6.6997.60 2 3.60 (3.1) LAPM 8 52 2 91 53 -4.0006 6.6997.60 2 3.60 (5.3) LAPM 23 182 8 299 163 4.030 5.0744-09 2 3.60 (5.3) LAPM 23 182 52 25 4.55106 8.6784-69 2 5.00 (5.3) LAPM <td>(200.200)</td> <td>LAPM</td> <td>22</td> <td>7286</td> <td>1007</td> <td>9096</td> <td>7998</td> <td>0.0000</td> <td>1.1341e-08</td> <td>7</td> <td>881.40</td>	(200.200)	LAPM	22	7286	1007	9096	7998	0.0000	1.1341e-08	7	881.40
BROWNALE LAPC 23 1471 300 195238 10704 00000 15416-07 6 1116-07 C02.00.0 LAPB 24 11534 10 177145 9708 0.0000 1.7318-07 7 2.91.30 BTI LAPG 22 67 1 62 33 -1.0000 6.0599-66 1 2.10 BT1 LAPB 8 55 2 91 53 -1.0006 6.0599-66 1 2.10 BT2 LAPB 29 7.2 8 614 38 0.03.39 90734-69 2 3.41 BT3 LAPC 23 182 290 153 4.0930 5.0744-69 2 4.00 (5.3) LAPG 23 182 7 113 60 4.0930 5.0744-69 2 5.00 (6.3) LAPG 23 182 7.710 7.45516 8.0754-99 2 7.10 (6.3)	()	LAPB	22	6656	1006	12252	10234	0.0000	6.4412e-00	7	863.28
DBM DAM 26 APR 26 APR 2600 12371-67 9 10130 CI LAPB 24 11534 11714 77125 22 250 FT LAPC 22 67 1 62 33 -1.0006 6.0598-66 1 2.10 C1) LAPM 8 55 2 91 53 -1.0006 6.0598-66 1 2.10 G.1) LAPR 10 102 5 164 87 2.1977 8.4332-69 2 3.01 BT LAPR 13 102 5 164 87 2.1977 8.4332-69 2 4.71 BT LAPR 23 158 7 113 66 40930 5.1746-69 2 7.0 G.2) LAPM 21 170 2 3.05 166 9.455106 8.0748-69 2 7.0 G.2) LAPM 21 170	BROWNALE.	LADC	22	14571	1000	105529	10204	0.0000	5.6100- 06	6	1116.20
LAPR 20 4/7 1004 4/2/3 2/31 0.0000 1.5/1-0/0 6 6/697 IT LAPC 23 1/34 1/11/4 9/986 A.0006 1.5/1-0/0 6 6/697 IT LAPC 23 1/11/4 1/11/4 201 2.10 IT LAPB 8 55 2 91 35 -1.0006 6.059%-66 1 2.10 IT LAPC 1/1 1/1 5 1/64 87 2.1877 8.4382-69 2 3.60 (3.1) LAPR 1/2 1/2 8 2/99 1/3 4.0930 5/074-69 2 4.01 (3.1) LAPR 23 1/2 8 2/99 1/3 4.0930 5/074-69 2 4.60 (3.1) LAPR 23 1/2 1/2 2/9 4.5106 2/1/3 4.090-0 2 3.06 C3.2) LAPR 21 1/9	BROWNALE	LAPC	23	14371	3	193328	107094	0.0000	3.01008-00	0	1110.20
LAPR 24 1134 1 17/14S 97M8 0.0000 1.9238-06 6 6092 C1.1 LAPR 8 5 2 91 33 -1.0000 4.072400 2 2.50 G1.1 LAPR 8 5 2 91 33 -1.0000 4.07540 2 2.50 G3.1 LAPR 19 102 5 164 87 2.1897 8.4382-09 2 3.60 G.5.1 LAPR 23 182 7 113 60 4.0930 5.0784-09 2 4.60 G.5.1 LAPR 23 182 7 113 60 4.0930 5.0784-09 2 4.710 G.5.1 LAPR 21 178 22 25.05 8.6784-09 2 4.710 G.5.1 LAPR 21 178 4.071 4.55166 8.7464-09 2 3.59 BT <thlapr< th=""> 21 178</thlapr<>	(200,200)	LAPM	26	4278	1004	45283	28113	0.0000	1./4/1e-0/	1	291.90
BTI LARC 22 67 1 62 33 -1.0006 6.43772-09 2 2.50 LAPB 8 55 2 91 53 -1.0006 6.6599-66 1 2.10 G.1) LAPC 20 72 8 61 83 0.0032 917316-09 2 2.30 G.1 LAPM 10 012 5 164 87 2.1577 8.3382-09 2 3.40 DT LAPC 13 13 60 40930 5.0744-09 2 4.60 C.3.0 LAPM 23 142 5 425 255 4.55106 8.678-09 2 7.0 G.3.1 LAPM 21 178 2 120 69 4.55106 8.374-09 2 3.60 G.3.1 LAPM 21 170 2.771 4.407520 2 3.60 G.3.1 LAPM 21 193 60 <th3< td=""><td></td><td>LAPB</td><td>24</td><td>11534</td><td>1</td><td>177145</td><td>97908</td><td>0.0000</td><td>1.9238e-06</td><td>6</td><td>669.27</td></th3<>		LAPB	24	11534	1	177145	97908	0.0000	1.9238e-06	6	669.27
LAPR 8 55 2 91 53 -1.0006 6.0599-06 1 2.10 BT2 LAPR 8 55 2 91 53 -1.0006 6.0599-06 1 2.01 BT2 LAPR 19 102 5 164 87 2.1877 8.4382-09 2 3.30 G.3.0 LAPR 13 163 2.1877 8.4382-09 2 4.30 G.3.0 LAPR 23 188 7 113 60 4.0930 6.051-09 2 4.71 G.3.0 LAPR 23 188 7 133 60 4.0930 6.051-09 2 5.06 G.3.1 LAPR 21 178 2 120 6 4.5106 8.774-09 2 3.30 G.3.1 LAPR 21 179 2 300 164 961.7152 4.004-09 2 3.30 G.3.1 LAPR 21	BT1	LAPC	22	67	1	62	33	-1.0000	4.3772e-09	2	2.50
LAPE 2 91 53 -1.0005 6.0599-60 1 2.01 (3.1) LAPK 19 102 5 164 87 2.1897 8.4382-69 2 3.60 (3.1) LAPK 19 102 5 164 87 2.1897 8.4382-69 2 6.10 (3.3) LAPK 23 182 7 113 6.00 4.0930 5.0744-69 2 4.60 (3.3) LAPK 23 286 7 113 6.00 4.0930 5.0744-69 2 4.60 (3.2) LAPK 21 178 2 120 69 4.5106 8.7340-69 2 3.80 (3.2) LAPK 21 179 2 3.90 164 961.7152 4.094-69 2 3.89 (5.2) LAPK 21 119 2 3.09 164 961.7152 4.094-69 2 3.59 (5.2) <t< td=""><td>(2,1)</td><td>LAPM</td><td>8</td><td>55</td><td>2</td><td>91</td><td>53</td><td>-1.0006</td><td>6.0599e-06</td><td>1</td><td>2.10</td></t<>	(2,1)	LAPM	8	55	2	91	53	-1.0006	6.0599e-06	1	2.10
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	8	55	2	91	53	-1.0006	6.0599e-06	1	2.01
IAD LAPN 10 102 5 164 87 2.1897 5.4382-09 2 3.60 IT1 LAPC 23 189 3 290 153 4.0930 5.0744-09 2 6.10 (5.3) LAPM 23 182 7 113 60 4.0930 5.0744-09 2 4.71 (5.3) LAPK 24 242 5 425 225 4.455106 8.0734-09 2 5.00 (5.2) LAPM 11 178 2 120 6 94.455106 8.0734-09 2 3.80 (5.2) LAPM 11 170 5 166 94 96.17152 4.004-09 2 3.80 (5.2) LAPM 11 19 2 309 164 96.17152 4.004-09 2 3.80 (5.2) LAPM 21 110 3.06 34 306.5000 5.1756.09 2 3.80	DT7	LARC	20	72	-	61	29	0.0226	0.0731a.00	2	2.50
(1).1 LAN 19 102 5 164 87 2.1897 8.4322.69 2 3.00 BT3 LAR 23 189 3 200 153 4.0930 6.1998.09 2 4.61 BT3 LAR 23 188 7 113 4.0930 6.1998.09 2 4.61 G.3 LAPB 23 188 7 113 60 4.0930 5.1798.09 2 7.0 G.3 LAPB 21 178 2 120 69 -455106 8.0778.09 2 3.60 G.3 LAPC 21 120 2 309 164 961.7152 4.0094.09 2 3.80 G.3 LAPC 22 342 6 211 117 0.2711 4.0155.09 2 3.80 G.3 LAPM 23 342 6 301 1000711 4.0175.24 4.0094.09 2 2.510	(2.1)	LAIC	10	102	6	164	50	0.0520	9.07310-09	2	2.50
LARG 19 102 5 164 87 2.1897 8.4382.69 2 3.41 (5.3) LARM 23 142 8 299 163 4.0930 6.0821.69 2 4.61 BT LARC 23 142 8 299 163 4.0930 5.0821.69 2 4.60 BT LARC 23 142 8 299 163 967.75 4.607.69 2 5.60 G.3.0 LARC 21 179 2 120 69 4.5106 8.774.69 2 3.80 G.3.1 LARC 21 119 2 309 164 961.7152 4.0045.09 2 3.80 G.3.1 LARM 19 33 6 276 146 0.27711 7.4025.09 2 8.80 G.3.1 LARM 23 1001 13 6.30 149 306.5000 5.2158.60 2 2.50 <t< td=""><td>(3,1)</td><td>LAPM</td><td>19</td><td>102</td><td>2</td><td>164</td><td>87</td><td>2.1897</td><td>8.43826-09</td><td>2</td><td>5.00</td></t<>	(3,1)	LAPM	19	102	2	164	87	2.1897	8.43826-09	2	5.00
BT3 LAPC 23 182 3 290 153 40930 6.0821-69 2 6.10 (5.3) LAPB 23 188 7 113 60 40930 5.0744-69 2 4.71 (3.2) LAPC 24 242 5 425 255 425 6.455106 8.073e-09 2 5.00 (3.2) LAPB 21 178 2 120 69 4.45106 8.734e-09 2 3.80 (3.2) LAPC 21 110 5 160 94 961.712 4.0078-09 2 3.80 (5.2) LAPB 21 110 5 160 941 9.0771 7.4025-09 2 10.00 (5.3) LAPM 24 101 3 68 11 305.500 6.146-09 2 2.750 (5.3) LAPM 24 101 3 58 338 305.500 6.146-09 2<		LAPB	19	102	5	164	87	2.1897	8.4382e-09	2	3.41
(5.3) LAPB 23 142 8 299 163 4.0930 5.0748-09 2 4.60 BT4 LAPB 23 158 7 113 60 4.0930 5.1798-09 2 4.71 BT4 LAPM 22 186 4 116 67 455.106 8.0678-09 2 4.86 G.3.0 LAPM 21 119 2 309 164 961.7152 4.0094-00 2 3.80 G.3.1 LAPB 21 119 2 309 164 961.7152 4.0094-00 2 8.80 G.2.1 LAPB 21 119 2 309 164 961.7152 4.0094-00 2 8.80 G.2.1 LAPB 24 120 3 6.30 349 306.500 5.318-00 2 2 2.018 G.2.1 LAPM 24 120 3 866 473 1.0000 5.318-00 <t< td=""><td>BT3</td><td>LAPC</td><td>23</td><td>189</td><td>3</td><td>290</td><td>153</td><td>4.0930</td><td>6.0821e-09</td><td>2</td><td>6.10</td></t<>	BT3	LAPC	23	189	3	290	153	4.0930	6.0821e-09	2	6.10
LAPE 23 158 7 113 60 4.0930 5.178e.09 2 4.711 (3.1) LAPR 124 242 5 425 165.106 8.078-c09 2 5.00 (3.2) LAPR 21 178 2 120 69 -45.5106 8.078-c09 2 3.80 (3.2) LAPM 119 2 309 164 961.7152 4.0094-c09 2 3.80 (5.2) LAPM 19 363 6 276 1.46 0.2771 7.465-c09 2 1.100 (5.3) LAPM 19 363 6 276 1.46 0.2771 7.465-c09 2 3.710 (5.3) LAPM 24 1019 8 1 1 306.5000 5.2158-c09 2 3.710 (5.3) LAPM 24 100 1.5 3 31 1.0000 8.238-c09 2 3.60 (5.3)	(5,3)	LAPM	23	142	8	299	163	4.0930	5.0744e-09	2	4.60
		LAPB	23	158	7	113	60	4.0930	5.1798e-09	2	4.71
	BT4	LAPC	24	242	5	425	225	-45 5106	2 6374e=09	2	7.10
LAPE LAPE 11 178 2 120 69 -45.3106 8.7340-60 2 4.46 BT5 LAPC 21 110 2 100 11 100 2 330 Col LAPB 21 119 2 300 164 961.7152 4.0054-60 2 330 Col LAPB 22 342 6 211 117 0.2711 4.0155-60 2 9.01 (5.2) LAPM 24 1019 8 6 301 166 0.2771 1.7405-60 2 9.761 BT7 LAPC 24 1201 13 630 349 306.500 4.271 0.00 2.158-60 2 2.575-0 2 3.60 (5.2) LAPM 24 110 3.58 31 1.0000 8.238-00 2 3.60 (5.2) LAPM 24 186 8 399 227 1.0000	(2.2)	LADM	27	186	1	116	67	45.5106	8.0678-00	2	5.60
LAPE 21 178 22 120 609 4430106 85.3582-69 2 4.88 (3.2) LAPB 21 119 2 309 164 961.7152 4.0094-699 2 3.02 BT6 LAPE 22 327 6 211 117 0.27711 4.0175-697 2 8.90 (5.2) LAPM 19 36.5 6 276 146 0.2771 1.075-697 2 9.61 BT7 LAPC 24 1201 13 6.30 349 306.5000 5.2158-697 2 7.10 (5.2) LAPM 24 1019 8 1 1 3005.500 6.2158-697 2 4.50 (5.2) LAPM 24 111 3 58 31 1.0000 8.884-09 2 5.00 (4.2) LAPM 24 180 8 399 227 -1.0000 9.4884-09 2 3.00	(5,2)	LAPM	22	180	4	110	67	-43.3100	8.00786-09	2	5.00
BTS LAPK 21 120 5 166 94 961.7152 3.582-69 2 3.80 (3.2) LAPM 21 119 2 300 164 961.7152 4.0094-69 2 3.29 (5.2) LAPK 22 22 76 6 211 117 0.2771 7.4025-69 2 1.80 (5.2) LAPK 22 342 6 301 166 0.2771 1.6798-69 2 37.10 (5.3) LAPK 24 1019 8 1 1 306.5000 5.2186-69 2 2.75.0 (5.3) LAPK 24 1095 6 958 538 306.5000 6.166-09 2 3.60 (5.2) LAPM 24 111 3 58 31 1.0000 8.684-69 2 7.00 (4.2) LAPM 24 186 8 399 27 -1.0000 9.648-09		LAPB	21	1/8	2	120	69	-45.5106	8.7340e-09	2	4.86
(3.2) LAPB 21 119 2 309 164 961.7152 4.0094-c90 2 3.59 BT6 LAPC 22 287 6 211 117 0.2771 4.0078-c90 2 8.90 C5.2) LAPM 19 363 6 216 116 0.2771 1.7678-c90 2 9.11 G5.3) LAPM 24 1019 8 1 1 306.5000 5.2158-c90 2 2.750 G5.3) LAPM 24 1019 8 1 1 306.5000 5.2158-c90 2 4.50 G5.2) LAPB 34 406 2 535 293 1.0000 8.084-c90 2 5.00 G5.2) LAPM 24 186 8 399 227 -1.0000 9.884-c90 2 5.00 G2.1 LAPB 10 136 4 313 220 -1.0000 9.884-c90 1	BT5	LAPC	21	120	5	166	94	961.7152	3.5582e-09	2	3.80
LAPE 21 119 2 309 164 961.7152 4.0078-09 2 3.29 (5.2) LAPM 19 363 6 276 146 0.2711 7.4075-09 2 11.00 BT7 LAPE 22 342 6 301 166 0.2771 1.6798-09 2 37.10 (5.3) LAPM 24 1019 8 1 1 306.5000 51288-09 2 2.750 (5.3) LAPM 24 1019 8 1 1 306.5000 6.116-09 2 2.618 (5.2) LAPM 24 111 3 58 31 1.0000 8.684-09 2 5.50 (4.2) LAPB 10 136 4 413 220 -1.0000 9.648-09 2 7.00 (4.2) LAPM 20 133 8 523 2.50 1.0000 1.9468-08 1 4.00 <	(3,2)	LAPM	21	119	2	309	164	961.7152	4.0094e-09	2	3.60
BT6 LAPC 22 287 6 211 117 0.2771 4.0178-09 2 8.90 (5.2) LAPM 22 342 6 301 166 0.2771 1.678-09 2 9.61 (5.3) LAPC 24 1019 8 1 1 306.5000 52.158-09 2 27.50 (5.3) LAPE 23 1095 6 958 538 306.5000 42.070-09 2 2.750 BT8 LAPC 34 230 1 1936 1054 10000 1.6860-09 2 4.50 (5.2) LAPK 24 111 3 58 31 1.0000 9.5832-09 2 5.00 (4.2) LAPK 24 186 4 413 220 -1.0000 9.5832-09 2 5.00 (2.2) LAPK 13 8 523 2.81 -1.0000 1.4469-08 1 4.00 </td <td></td> <td>LAPB</td> <td>21</td> <td>119</td> <td>2</td> <td>309</td> <td>164</td> <td>961.7152</td> <td>4.0094e-09</td> <td>2</td> <td>3.29</td>		LAPB	21	119	2	309	164	961.7152	4.0094e-09	2	3.29
	BT6	LAPC	22	287	6	211	117	0.2771	4.0175e=09	2	8.90
	(5.2)	LAPM	10	363	6	276	146	0.2771	7.4625e-09	2	11.00
LAPC 22 342 00 301 100 0.2/11 10.398c-09 2 301 (5.3) LAPB 24 1019 8 1 1 306.5000 5.218c-09 2 2.750 (5.2) LAPC 34 230 10 1936 1054 1.0000 8.1660-09 2 4.50 (5.2) LAPB 34 406 2 535 293 1.0000 8.2328-09 2 3.60 (4.2) LAPB 34 406 2 535 293 1.0000 9.8534-09 2 5.90 (4.2) LAPB 34 406 2.20 -1.0000 9.4530-06 1 4.17 BT10 LAPC 16 169 3 1 1 -1.0000 6.4382-07 1 5.50 (2.2) LAPM 20 133 8 523 2.81 -1.0000 3.4497 3.360 3.300 (2.3)	(3,2)	LADD	22	242	6	201	140	0.2771	1.40250-00	2	0.61
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPD	22	342	0	301	100	0.2771	1.07986-09	2	9.01
(5.3) LAPM 24 109 8 1 1 306.5000 6.2370-69 2 27.50 BT8 LAPC 34 230 1 1936 1054 1.0000 6.2328-09 2 4.50 (5.2) LAPB 34 406 2 535 293 1.0000 8.2328-09 2 3.60 BT9 LAPC 24 186 8 399 277 -1.0000 9.848-09 2 7.00 (4.2) LAPM 24 186 8 399 227 -1.0000 9.5438-09 2 5.90 (2.2) LAPM 20 133 8 523 281 -1.0000 6.4382-07 1 5.90 (2.3) LAPM 20 113 436 233 -1.0000 3.5406-06 1 4.90 (5.3) LAPK 20 113 5 2588 0.8249 7.5306-09 2 3.90 <t< td=""><td>B17</td><td>LAPC</td><td>24</td><td>1201</td><td>13</td><td>630</td><td>349</td><td>306.5000</td><td>5.2158e-09</td><td>2</td><td>37.10</td></t<>	B17	LAPC	24	1201	13	630	349	306.5000	5.2158e-09	2	37.10
	(5,3)	LAPM	24	1019	8	1	1	306.5000	4.2470e-09	2	27.50
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	23	1095	6	958	538	306.5000	6.1416e-09	2	26.18
(5.2) LAPM 24 111 3 58 31 1.0000 2.8238-09 2 3.60 BT9 LAPC 24 220 3 866 473 -1.0000 9.6848-09 2 8.07 (4,2) LAPM 24 186 8 399 227 -1.0000 9.5430-06 1 4.17 BT10 LAPC 16 169 3 1 1 -1.0000 9.5430-06 1 4.00 (2.2) LAPB 14 101 3 436 233 -1.0000 1.4916-08 1 4.00 (2.3) LAPB 14 101 3 436 233 -1.0000 1.4916-08 1 4.00 (5.3) LAPB 14 101 3 436 233 -1.0000 3.545e-10 2 3.76 (5.3) LAPB 26 124 2 107 131 6.1881 2.8309-09 2 3.2	BT8	LAPC	34	230	1	1936	1054	1.0000	1.1660e-09	2	4.50
	(5.2)	LAPM	24	111	3	58	31	1.0000	8 2328e=09	2	3.60
BT9 LARC 24 220 3 866 473 -1.0000 9.4884e-09 2 5.00 (4.2) LAPM 24 186 8 399 227 -1.0000 9.5484e-09 2 5.90 LAPB 10 136 4 413 220 -1.0000 9.5430e-06 1 4.17 BT10 LAPC 16 169 3 1 1 -1.0000 6.4382e-07 1 5.50 (2.2) LAPB 14 101 3 436 233 -1.0000 3.5405-06 1 2.99 BT11 LAPC 29 155 4 93 50 0.8249 7.3506-09 2 3.70 (5.3) LAPM 20 119 2 257 141 0.8249 7.3506-09 2 3.20 (5.3) LAPM 20 119 2 257 141 0.8249 7.3506-09 2 3.20	(3,2)	LADD	24	406	2	535	202	1,0000	2 61200 10	2	8.07
B19 LAPC 24 20 5 860 41.3 -1.0000 9.8884-09 2 7 7.00 LAPB 10 136 4 413 220 -1.0000 9.5430e-06 1 4.17 BT10 LAPC 16 169 3 1 1 -1.0000 5.432e-07 1 5.50 (2,2) LAPB 14 101 3 436 233 -1.0000 3.545e-06 1 4.00 (5.3) LAPB 14 101 3 436 233 -1.0000 3.545e-09 2 3.90 (5.3) LAPB 26 124 2 109 58 0.8249 7.3560e-09 2 3.26 3.124 BYRDSPHR LAPC 24 105 5 275 151 -4.6833 2.0140e-07 1 2.10 2.30 (3.2) LAPB 18 69 4 138 78 -4.6833 2.	D.770	LAPB	34	400	2	333	293	1.0000	2.0120e-10	2	8.07
	B19	LAPC	24	220	- 3	800	4/3	-1.0000	9.8684e-09	2	7.00
	(4,2)	LAPM	24	186	8	399	227	-1.0000	9.1683e-09	2	5.90
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	10	136	4	413	220	-1.0000	9.5430e-06	1	4.17
	BT10	LAPC	16	169	3	1	1	-1.0000	6.4382e-07	1	5.50
	(2.2)	LAPM	20	133	8	523	281	-1.0000	1 4916e=08	1	4.00
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(2,2)	LADD	14	101	2	126	222	1.0000	3 54050 06	1	2.00
	DELL	LAID	14	101	5	430	233	=1.0000	3.34036=00	1	2.99
(5,3) LAPM 20 119 2 257 141 0.8249 7.5360e-09 2 3.30 BT12 LAPC 23 1137 5 2588 1425 6.181 2.8306e-09 2 3.376 (5.3) LAPM 20 119 2 257 141 0.8249 7.5360e-09 2 3.30 (3.2) LAPB 27 1478 6 2340 1331 6.1881 5.9686e-11 5 31.24 BYRDSPHR LAPC 24 105 5 275 151 -4.6833 5.2070-09 2 3.20 (3.2) LAPM 18 69 4 138 78 -4.6833 2.0140e-07 1 2.30 (2.2) LAPB 16 40 4 462 250 0.0000 4.7698e-06 1 1.40 (2.2) LAPB 16 40 4 462 250 0.0000 4.7698e-06 1 <td>BIII</td> <td>LAPC</td> <td>29</td> <td>155</td> <td>4</td> <td>93</td> <td>50</td> <td>0.8249</td> <td>1.9859e-11</td> <td>5</td> <td>4.90</td>	BIII	LAPC	29	155	4	93	50	0.8249	1.9859e-11	5	4.90
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(5,3)	LAPM	20	119	2	257	141	0.8249	7.5360e-09	2	3.90
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	26	124	2	109	58	0.8249	2.3365e-10	2	3.76
	BT12	LAPC	23	1137	5	2588	1425	6.1881	2.8309e-09	2	32.60
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(5.3)	LAPM	20	119	2	257	141	0.8249	7.5360e-09	2	3.90
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(4,4)	LAPB	27	1478	6	2340	1331	6 1881	5.9686e-11	5	31.24
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	DVDDCDUD	LADG	27	1470	6	2340	1551	0.1001	5.0000-11	2	3.24
	BTRDSPHK	LAPC	24	105	5	275	151	-4.0855	5.2207e-09	2	3.20
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(3,2)	LAPM	18	69	4	138	78	-4.6833	2.0140e-07	1	2.30
		LAPB	18	69	4	138	78	-4.6833	2.0140e-07	1	2.11
	CLUSTER	LAPC	28	50	1	255	134	0.0000	7.9137e-10	2	1.70
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(2.2)	LAPM	16	40	4	462	250	0.0000	4.7698e-06	1	1.40
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	16	40	4	462	250	0.0000	4.7698e-06	1 1	1.39
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	COOLITANS	LADC	2	10002	1	201527	162541	0.0000	0.7970	6	250.40
	COULIANS	LAPC		10002		301327	102341	0.0000	0.7870	0	239.40
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(9,9)	LAPM	2	10002	0	304468	163652	0.0000	0.3083	0	243.40
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		LAPB	2	10002	0	303163	163057	0.0000	4.2353e-01	6	222.34
	CUBENE	LAPC	16	116	1	402	227	0.0000	4.7280e-06	1	3.60
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	(2.2)	LAPM	16	185	9	571	316	0.0000	7.7815e-07	1	5.20
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		LAPB	19	165	12	399	216	0.0000	6.5847e-07	1	4.25
	DIXCHING	LAPC	20	502	5	303	212	0.0000	5.9562e-11	5	17.80
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(10.5)	LADM	27	572	6	212	114	0.0000	7.8014-11	5	17.00
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(10,5)	LAPM	27	008	0	212	114	0.0000	/.8014e-11	2	17.70
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	27	559	5	509	282	0.0000	8.56/3e-11	5	15.30
	EIGENA2	LAPC	21	56	2	62	33	0.0000	8.6855e-09	2	2.00
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(6,3)	LAPM	22	51	10	1	1	0.0000	3.9460e-09	2	1.80
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	22	51	10	1	1	0.0000	3.9460e-09	2	1.66
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	EIGENA?	LAPC	30	2287	8	46	25	0.0000	5.1280e-11	5	132 70
	(110.55)	LADM	26	2200	ő	105	2.5	0.0000	9.7565-11	5	172.40
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(110,55)	LAPM	20	3209	1	105	20	0.0000	6./303e-11	3	1/2.40
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	20	3254	11	195	108	0.0000	7.6520e-09	2	149.65
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	EIGENB	LAPC	37	5482	1008	9041	8655	0.0000	2.5229e-11	7	302.10
LAPB 32 5767 1013 36200 7204 0.0000 1.9590e-10 7 253.30 EIGENB2 LAPC 30 86 11 46 25 2.0000 6.1197e-11 5 2.60 (6.3) LAPM 22 115 11 67 41 2.0000 4.271e-09 2 4.00 LAPB 29 134 10 185 102 2.0000 9.1710e-12 5 3.93 EIGENB2 LAPC 30 334 12 44 24 0.4473 6.3877e-11 5 19.30 (110.55) LAPM 30 278 12 101 60 0.4473 9.7328e-11 5 14.70	(110,110)	LAPM	35	6473	1014	8987	8113	0.0000	6.2406e-11	7	333.40
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$, .,	LAPB	32	5767	1013	36200	7204	0.0000	1.9590e-10	7	253.30
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	EIGEND2	LARC	20	86	10.5	46	25	2,0000	6 11070 11	5	2.60
(0.5) LAPM 22 115 11 07 41 2.0000 4.2271e-09 2 4.00 LAPB 29 134 10 185 102 2.0000 9.1710e-12 5 3.93 EIGENB2 LAPC 30 334 12 44 24 0.4473 6.3877e-11 5 19.30 (110,55) LAPM 30 278 12 101 60 0.4473 9.7328e-11 5 14.70	EIGENB2	LAPC	20	00		40	2.5	2.0000	0.119/0-11	5	2.00
LAPB 29 134 10 185 102 2.0000 9.1710e-12 5 3.93 EIGENB2 LAPC 30 334 12 44 24 0.4473 6.3877e-11 5 19.30 (110,55) LAPM 30 278 12 101 60 0.4473 9.7328e-11 5 14.70	(0,3)	LAPM	22	115	11	67	41	2.0000	4.2271e-09	2	4.00
EIGENB2 LAPC 30 334 12 44 24 0.4473 6.3877e-11 5 19.30 (110,55) LAPM 30 278 12 101 60 0.4473 9.7328e-11 5 14.70		LAPB	29	134	10	185	102	2.0000	9.1710e-12	5	3.93
(110,55) LAPM 30 278 12 101 60 0.4473 9.7328e-11 5 14.70	EIGENB2	LAPC	30	334	12	44	24	0.4473	6.3877e-11	5	19.30
	(110.55)	LAPM	30	278	12	101	60	0.4473	9.7328e-11	5	14.70
	(10,55)	LAIM		270	12	101	00	3.777.5	1 .15200-11	1	1 17.10

Table 1: Results of CUTEr collection problems

								Table 1 - continu	ied from p	revious page
Problem	Method	it.ext	it.int	search	#£	#∇ <u>L</u>	$f(\bar{x})$	$ h(\tilde{x}) $	exit	time
(n, m)	LADD	20	207	16	101	(0)	0.4472	7.5542.11	6	(seconds)
EIGEND2	LAPB	28	297	10	101	60	0.4473	7.5542e-11	5	13.91
(420.210)	LAPM	30	622	9	93	50	0.1924	6.6859e-11	5	128.70
	LAPB	29	565	8	99	56	0.2356	2.6172e-11	5	86.42
EIGENB2	LAPC	31	1320	4	42	23	0.0892	4.7504e-11	5	2030.30
(930,465)	LAPM	28	916	5	99	56	0.1466	2.1593e-11	5	1043.10
FIGENBCO	LAPB	20	1140	10	62	33	0.0496	5.220/e-11 8.7623e-09	2	6.30
(6.3)	LAPM	12	133	5	83	49	1.0000	9.6251e-07	1	4.60
	LAPB	28	153	5	191	102	1.0000	2.9474e-11	5	4.64
EIGENBCO	LAPC	31	720	8	44	24	0.2240	4.1793e-11	5	49.20
(110,55)	LAPM	28	841	8	97	55	0.2238	4.8099e-11	5	47.80
FIGENBCO	LAPC	20	1274	11	287	52	0.2240	9.2900e-11	5	1220.60
(650,325)	LAPM	30	1371	9	46	25	0.2061	5.6484e-11	5	936.30
	LAPB	28	1429	7	93	50	0.2119	6.7913e-11	5	552.24
EIGENC	LAPC	6	10387	3	150545	79834	0.0000	0.8230	6	371.40
(30,30)	LAPM	30	5718	8	952	523	0.0000	4.8817e-11	5	210.90
FIGENC	LAPB	1	5189	0	529 425882	292	0.0000	5.8801e-11 15.0993	3	188.45
(462,462)	LAPM	2	10002	1	405165	214128	0.0000	0.7314	8	5682.70
	LAPB	2	10002	1	405674	214625	0.0000	7.3071e-01	6	3157.40
EIGENC2	LAPC	24	247	13	56	30	0.0000	1.2093e-09	2	10.50
(30,15)	LAPM	19	217	8	71	43	0.0000	7.6356e-09	2	8.30
FIGENC2	LAPB	24	5280	2	59	37	0.0000	1.1620e-09	2	8.45
(462.231)	LAPM	29	3721	9	101	54	0.0000	7.6473e-11	5	847.90
	LAPB	31	5399	5	177	98	0.0000	2.3865e-11	5	1014.90
EIGENC2	LAPC	3	5046	0	94486	49852	2.0259	3.5891	8	4379.70
(650,325)	LAPM	8	6647	5	41443	21890	0.6305	1.0057	8	3839.80
FLEC	LAPB	4	59/1	2	95684	50539	2.4435	3.1453e+00	8	4039.70
(75.25)	LAPC	30	98	8	97	58	243.8133	2.2824e-12	5	4.40
(,)	LAPB	30	104	8	91	52	243.8131	1.5187e-11	5	8.11
ELEC	LAPC	31	341	7	44	24	1055.2000	9.0312e-11	5	25.80
(150,50)	LAPM	33	137	5	93	56	1055.2000	3.3841e-11	5	9.30
ELEC	LAPB	33	137	5	93	56	1055.1829	3.3841e-11	5	16.62
(300 100)	LAPC	52	1915	1009	1066	3068	4448.4000	1.4270e-15	7	218 20
(500,100)	LAPB	53	1236	1013	2238	3862	4448.3593	4.0000e-15	7	236.42
GENHS28	LAPC	20	133	2	1	1	0.3636	9.7362e-09	2	4.90
(5,3)	LAPM	27	176	4	57	36	0.3637	2.0579e-10	2	5.80
CENILS28	LAPB	25	161	1	207	110	0.3636	5.3448e-10	2	8.74
(10.8)	LAPC	29	212	9	247	133	0.9272	4.4338e-11	5	7.00
(,-)	LAPB	27	203	7	279	152	0.9273	8.3576e-11	5	10.00
GENHS28	LAPC	30	207	4	210	119	1.4816	8.4960e-11	5	7.10
(15,13)	LAPM	28	229	13	214	121	1.4815	8.4045e-11	5	7.10
COTTED	LAPB	29	212	1	284	156	1.4815	5.7145e-11	5	11.28
(2.2)	LAPC	25	181	1	1287	555	0.0000	2.2755e-09	2	5.00
(=,=)	LAPB	25	164	2	331	175	0.0000	2.1208e-09	2	7.36
GRIDNETE	LAPC	30	359	7	169	94	39.6060	7.2683e-11	5	15.00
(60,36)	LAPM	29	331	4	474	260	39.6057	8.1700e-11	5	12.10
CRIDNETE	LAPB	31	374	7	399	221	39.6060	3.2291e-11	5	22.77
(180 100)	LAPC	31	629	4	360	203	50.6038	4.9382e-11	5	35.80
(100,100)	LAPB	30	622	9	333	182	50.6057	9.6051e-11	5	51.99
GRIDNETE	LAPC	34	1142	10	703	391	75.5583	9.3200e-12	5	906.20
(612,324)	LAPM	34	1176	8	152	87	75.5566	7.4148e-12	5	500.30
GRIDNETE	LAPB	31	120/	9	303	202	87 2658	9.7163e-11	5	1846.00
(924, 484)	LAPM	35	1339	7	1001	552	87.2649	4.2362e-11	5	1308.90
	LAPB	33	1713	7	1183	655	87.2678	2.5331e-11	5	1711.80
GRIDNETH	LAPC	31	714	7	504	272	57.0635	8.1426e-11	5	132.80
(264,144)	LAPM	30	859	9	900	494	57.0643	3.2437e-11	5	66.10
GRIDNETU	LAPB	31	/10	13	500 871	270	37.0032	0.385/e-11	5	95.05
(924,484)	LAPM	34	1284	7	578	318	87.2757	6.8907e-11	5	1291.50
	LAPB	38	1523	7	763	430	87.2772	1.7390e-12	5	1765.50
GRIDNETH	LAPC	6	465	1	16151	8508	92.6631	0.4236	8	3879.00
(1984,1024)	LAPM	11	1272	3	1556	814	115.5468	3.9415e-04	8	3605.10
HEART6	LAPB	3	14783	0	289542	156192	0.0000	0.2839	6	409.40
(6,6)	LAPM	2	10002	0	297083	159926	0.0000	0.0762	6	238.60
	LAPB	2	10002	0	293688	158383	0.0000	7.5837e-02	6	426.52
HEART8	LAPC	26	2117	6	3238	1773	0.0000	3.0204e-09	2	69.40
(8,8)	LAPM	29	2511	1004	4666	5543	0.0000	4.1923e-11	7	69.50
HIMMETDA	LAPB	2/	2141	1003	20988	55/8 708	0.0000	8.9203e-11	3	115.94
(2,2)	LAPM	21	64	11	63	39	0.0000	7.0226e-09	2	2.00
	LAPB	21	64	11	63	39	0.0000	7.0226e-09	2	3.73
HIMMELBC	LAPC	19	34	2	121	64	0.0000	8.5681e-09	2	1.30
(2,2)	LAPM	17	39	3	134	76	0.0000	6.8066e-08	1	1.20
HIMMEI BE	LAPB	22	- 39 - 91	3	1.54	70	0.0000	6.8006e-08	2	2.55
THANHELDE	LAIC			L '	127	17	0.0000	0.70000-09	ontinued	on peyt page

Two new	augmented	Lagrangian	algorithms	with	quadratic	penalty	for equality	problems
1.00	aaginentea	Bugrungrun	angomanno		quadratic	Penany	for equality	prooreino

								Table 1 - continu	ied from p	previous page
Problem	Method	it.ext	it.int	search	#£	#∇ <u>_</u> L	$f(\bar{x})$	$ h(\bar{x}) $	exit	time (manuals)
(1, m)	LAPM	20	99	3	125	66	0.0000	9.7309e-09	2	3.10
	LAPB	24	122	5	111	59	0.0000	1.5097e-09	2	6.61
HS6	LAPC	18	56	0	54	29	0.0000	3.3369e-09	2	2.40
(2,1)	LAPM	8	54	5	77	46	0.0000	2.2827e-06	1	3.97
HS7	LAPC	33	120	1	255	134	-1.7321	7.6324e-09	2	4.20
(2,1)	LAPM	28	59	0	125	66	-1.7321	5.0277e-09	2	2.20
HS8	LAPD	19	39 86	10	240	132	-1.0000	2.8202e-08	2	2.70
(2,2)	LAPM	18	98	2	121	64	-1.0000	5.7127e-09	2	2.80
1100	LAPB	18	98	2	121	64	-1.0000	5.7127e-09	2	5.48
(2.1)	LAPC	23	45	0	58	200	-0.4999	8.0977e-09	2	1.50
(=,-)	LAPB	23	45	0	58	31	-0.4999	8.0977e-09	2	2.86
HS11	LAPC	25	55	12	124	77	-8.4985	3.4970e-09	2	2.00
(2,1)	LAPM	21	41	4	1	1	-8.4985	3.0428e-09 3.0428e-09	2	3.12
HS26	LAPC	24	35	0	1	1	0.0001	1.4063e-09	2	1.40
(3,1)	LAPM	19	101	3	117	62	0.0002	4.4175e-09	2	3.70
11007	LAPB	24	118	4	1	1	0.0003	6.0295e-10	2	7.41
HS27 (3.1)	LAPC	22	106	5	69	42	0.0401	9./152e-09 1.4824e-09	2	4.30
(3,1)	LAPB	24	106	5	69	42	0.0400	1.4824e-09	2	7.64
HS28	LAPC	21	71	8	65	40	0.0001	2.9581e-09	2	2.70
(3,1)	LAPM	21	69	2	125	66 130	0.0000	9.1723e-09	2	2.70
HS39	LAPD	24	220	3	866	473	-1.0000	9,8684e-09	2	7.40
(4,2)	LAPM	24	186	8	399	227	-1.0000	9.1683e-09	2	5.80
	LAPB	10	136	4	413	220	-1.0000	9.5430e-06	1	8.44
HS40	LAPC	15	66	6	369	198	-0.2500	1.2999e-06	1	2.70
(4,5)	LAPB	22	95	0	355	195	-0.2500	8.9002e-09	2	5.83
HS42	LAPC	23	80	4	1	1	13.8579	8.7838e-10	2	3.00
(4,2)	LAPM	9	54	2	174	96	13.8579	7.4100e-06	1	1.80
11546	LAPB	9	54	2	174	96	13.8579	7.4100e-06	1	3.27
(5,2)	LAPM	21	81	1	223	118	0.0001	5.4540e-09	2	2.50
	LAPB	19	75	1	237	125	0.0001	6.8985e-09	2	4.30
HS47	LAPC	23	330	6	393	219	-0.0001	2.2207e-09	2	13.40
(5,3)	LAPM	23	268	7	122	59 70	-0.0002	5.7752e-09	2	8.70
HS48	LAPC	29	79	1	99	56	0.0000	5.3881e-11	5	4.20
(5,2)	LAPM	23	69	1	227	120	0.0000	5.1150e-09	2	2.20
11640	LAPB	19	67	1	62	33	0.0000	7.5528e-09	2	4.60
(5.2)	LAPC	20	150		215	114	0.0001	8.1800e-10	2	4.80
	LAPB	22	152	0	215	114	0.0000	8.1800e-10	2	8.45
HS50	LAPC	20	101	3	345	182	0.0000	3.4143e-09	2	3.80
(5,3)	LAPM	21	106		226	125	0.0000	3.6152e-09 3.7896e-09	2	6.11
HS51	LAPC	23	138	2	56	30	0.0000	3.7935e-09	2	5.40
(5,3)	LAPM	22	146	3	339	179	0.0000	3.3365e-09	2	4.30
11652	LAPB	23	169	8	604	325	0.0000	5.1640e-09	2	9.04
(5.3)	LAPC	25	202	8	124	71	5.3267	7.1856e-09	2	6.50
	LAPB	26	212	9	156	83	5.3267	5.4791e-10	2	11.67
HS56	LAPC	20	95	2	117	62	-3.4560	5.8923e-09	2	3.70
(7,4)	LAPM	19	8/	8	195	40	-3.4560	5./648e-09 9.8270e=09	2	5.23
HS61	LAPC	27	82	1	113	60	-143.6461	1.7032e-09	2	2.90
(3,2)	LAPM	20	60	1	109	58	-143.6461	3.1707e-09	2	1.90
11677	LAPB	20	60	1	109	58	-143.6461	3.1707e-09	2	3.46
HS// (5.2)	LAPC	20	253	5	217	121	0.2415	8.2504e-10 2.9626e-09	2	7.60
(LAPB	25	286	3	443	237	0.2415	1.2356e-09	2	15.34
HS78	LAPC	21	83	5	56	30	-2.9197	5.6326e-09	2	3.10
(5,3)	LAPM	20	70	6	56	30	-2.9197	9.2516e-09	2	2.10
HS79	LAPC	19	173	10	246	135	0.0788	4.9604e-09	2	6.60
(5,3)	LAPM	22	158	2	1	1	0.0788	4.1727e-09	2	4.90
Marcar ND	LAPB	21	154	2	62	33	0.0788	8.2467e-09	2	8.77
(7.2)	LAPC	21	613	6	245	135	680.6301	7.3892e-12	5	15.80
· · · · /	LAPB	25	579	4	449	246	680.6301	4.8668e-11	5	27.07
HS111LNP	LAPC	31	417	6	101	54	-45.8472	1.8106e-11	5	19.60
(10,3)	LAPM	30	595	13	402	218	-47.7595	4.0119e-11	5	20.90
HYDCAR6	LAPD	2/	10002	0	372113	198242	0.0000	0.3303	6	417.20
(29,29)	LAPM	2	10002	0	373702	198455	0.0000	0.3653	6	350.60
	LAPB	2	10002	0	373952	198927	0.0000	4.3256e-01	6	604.91
HYDCAR20 (99.99)	LAPC	2	10002	0	372668	198343	0.0000	0.4749	6	688.00
(77,77)	LAPB	2	10002	1	363907	193500	0.0000	4.3772e-01	6	917.00
HY,CIR	LAPC	23	63	7	115	61	0.0000	3.6847e-09	2	2.20
(2,2)	LAPM	21	66	4	119	63	0.0000	2.9855e-09	2	1.90

InserInserinser <th< th=""><th></th><th></th><th></th><th></th><th></th><th></th><th></th><th></th><th>Table 1 - continu</th><th>ied from p</th><th>revious page</th></th<>									Table 1 - continu	ied from p	revious page
(a, b) (a) (b) (c) (c)<	Problem	Method	it.ext	it.int	search	#£	#∇ <u>L</u>	$f(\bar{x})$	$ h(\bar{x}) $	exit	time
LIKEVELI LARC 21 27 2 27 33 00000 15145-00 2 17.0 (100.96) LARM 19 138 11 99 37 0.0000 6.316-61 2 13.5 (100.97) LARM 21 13 11 99 37 0.0000 6.316-61 2 13.5 (100.97) LARM 23 11.0 3 44 99 7.0 0.0000 6.316-61 2 7.336 (100.1) LARK 33 4.15 6 400 2.54 2.756-60 7.256 7.976-60 2 12.60 (100.2) LARK 30 346 9 200 156 2.7566 3.266 1 2.76 1.260 1.2 1.260 (100.2) LARK 30 50 17 2.7566 3.266 1 2.6 1.1 5.0 1.260 1.2 1.60 1.160 1.1 5.0	(<i>n</i> , <i>m</i>)	LADD	21	60	0	177	00	0.0000	1 3446a 00	2	(seconds)
LUNDE LAPR 19 128 11 99 37 0.0000 4.579-10 2 7.30° LIKVLII LAPR 21 138 11 99 37 0.0000 6.5146-10 2 135 LIKVLIZ LAPC 21 100 1 4 40 25 4976-00 2 135 LIKVLIZ LAPC 20 12 46 25 7566 47976-11 5 1236 LIKVLIZ LAPB 25 379 5 720 394 27566 47986-11 5 124 (10.2) LAPB 25 379 6 63 19 27366 37986-11 5 2276 (10.2) LAPB 23 376 6 63 9 273763 73708-11 5 424 (10.2) LAPB 23 377 637703 5398-110 5 2340 (10.4) 24 377	LUKVLEI	LAPC	21	176	2	52	28	0.0000	2.5145e-09	2	3.39
LAPB 200 188 11 54 57 0.0000 6.3154-00 2 1.535 (100.98) LAPM 21 164 7 58 57 0.0000 3.545-00 2 3.430 (100.70 LAPM 23 415 6 450 25 64.1597 7.5566 7.7996-11 5 1.130 (100.71 LAPM 23 427 5 111 59 27.5566 7.3996-11 5 1.160 (100.71 LAPM 23 427 5 111 59 27.5566 9.39878-11 5 12.160 LIKNLES LAPE 23 377 6 63 39 60.13 377.701 7.3888-09 2 13.160 LIKNLES LAPE 23 373 100 161 641 64 637.700 2.3888-00 2 13.156 LIKNLES LAPE 23 373 100 161 641 <th< td=""><td>(100,98)</td><td>LAPM</td><td>19</td><td>126</td><td>15</td><td>59</td><td>37</td><td>0.0000</td><td>4.7591e-10</td><td>2</td><td>7.30</td></th<>	(100,98)	LAPM	19	126	15	59	37	0.0000	4.7591e-10	2	7.30
LIKYLEI LACC 21 210 3 44 7 63235 44796-09 2 81.00 U0009/% LAW 24 145 6 25 604.1987 4.0972.11 5 1345.00 URVLEI LAW 25 379 5 720 394 27.5866 4.0986.10 2 10.4 UND_21 LAW 23 379 5 720 394 27.5866 4.0986.10 5 12.0 UND_21 LAW 20 364 9 200 15.0 27.5866 5.3496.11 5 2.27.6 UNVLE3 LAW 24 367 6 6.1 21 37.731 7.7386.11 5 4.23.0 UNVLE4 LAW 23 366 6.37.703 5.3496.11 5 4.30 UNVLE5 LAW 30 371 6 6.37.703 6.248.211 5 4.30 UNVLE5 LAW 30 371<		LAPB	20	138	11	59	37	0.0000	6.8164e-10	2	13.55
(1000.98) LAVM 21 164 7 59 17 00000 3.348.09 2 343.30 (100.1) LAVR 23 415 6 450 24.5 25.566 7.798.11 5 14.50 (100.2) LAVR 23 427 5 111 59 27.566 9.198.69 2 14.60 (100.2) LAVR 23 427 5 111 59 27.566 9.198.69 2 16.60 (100.2) LAVR 30 316 6 30 9 60.10 9 25.566 9.198.69 2 10.60 (100.2) LAVR 30 346 61 30 377.011 7.3888.69 2 13.60 (100.2) LAVR 32 378 100 100 101 30.6 601.97.00 2.09 2 17.15 LKVLE3 LAVR 52 537 100 <th103< th=""> 100 1013</th103<>	LUKVLE1	LAPC	21	210	3	54	29	6.2325	4.9760e-09	2	831.00
LUKVLE LAPK 20 100<	(1000,998)	LAPM	21	164	7	59	37	0.0000	3.5483e-09	2	346.30
Linoz LAW 25 4:15 6 4:50 2:54 9:7566 7:7576 7:75 7:756 7:7566 7:75761 7:757 7:756 7:7576 7:757 7:756 7:7576 7:757	LUKVLE2	LAPB	24	1/0	12	54	34	0.0000	3.31408-09	2	378.30
LAPE 25 379 5 730 394 27.886 49.055,10 2 21.94 (50.2) LAPC 23 383 3 50 27 27.586 2.345,400 2 12.00 (100.1) LAPC 23 383 10 23.586 2.345,400 2 12.00 (100.2) LAPA 24 307 6 63 19 64.197 3.368,600 2 34.10 (100.2) LAPA 23 305 6 46 25 377.011 8.188,811 5 9.03 LUKVLED LAPC 20 232 10 4.20 2.377.015 8.185,811 5 9.04 (09.4) LAPC 20 232 10 10.20 6.075,5300 2.448,615 7 4.00 (10.4) LAPC 23 244 10 101 111 31.66 6.055,5000 2.449,61 7 4.00 (10.4)	(100.2)	LAPM	28	415	6	450	245	27.5866	7.7980e-11	5	14.50
LIKVELES LAPA 23 427 5 111 99 27.556 0.3196-00 2 1.60 UNULL LAPA 23 888 3 50 27 27.5566 0.3414-00 2 1.60 UNULL LAPB 26 420 111 380 210 27.5566 0.3416-00 2 1.70 1.8 1.858 1.70 1.8 1.858 1.9 1.0 1.1 1.3 1.3 1.0 1.1 1.3 1.3 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 <th< td=""><td>()</td><td>LAPB</td><td>25</td><td>379</td><td>5</td><td>720</td><td>394</td><td>27.5866</td><td>4.9656e-10</td><td>2</td><td>21.94</td></th<>	()	LAPB	25	379	5	720	394	27.5866	4.9656e-10	2	21.94
(9.0.) LAPA 23 388 3 50 27 22.5866 2.544b-09 2 1.260 LURVLE LAPR 23 57 9 6 50 1.09 23.586 9 7 22.76 LURVLE LAPR 23 37 8 67 44 941.987 3.3886-07 2 34.96 UKVLE LAPR 23 306 9 220 121 377.701 7.3886-11 5 9.40 UKVLE LAPR 23 288 4 63 39 377.701 7.9282-69 2 2.13 (9A.4) LAPR 32 284 4 63 59 607.623 11855 15 2.13 4 155 151 115 115 115 115 115 115 125 115 125 115 126 109.99 109 1073 302 100 113 146 625.8000 127.15	LUKVLE3	LAPC	23	427	5	111	59	27.5866	9.3198e-09	2	16.60
LAPIE 2.0 4.00 11 3.00 7.10 2.1386 9.94 Ac.11 3.2 2.10 (100.2) LAPB 3.2 377 8 67 41 0.94 JBS7 3.3386-00 2 3.94 (100.2) LAPE 2.2 317 8 67 41 0.94 JBS7 3.1316-00 2.1 3.94 (104.1) LAPE 2.2 317 6 4.6 3.9 3.77.1031 7.1238-11 5 9.40 (0,4) LAPE 2.3 2.83 1.00 4.4 3.00 6.037.1000 6.0126-11 5 4.23.0 (0,4) LAPE 2.9 1.05 1.054 3.06 6.037.000 6.1368.000 2.37.0 1.5 4.00 (0,4) LAPM 2.2 1.05 1.054 3.06 6.037.000 6.037.000 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 6.037.00 <t< td=""><td>(50,2)</td><td>LAPM</td><td>23</td><td>388</td><td>3</td><td>50</td><td>27</td><td>27.5866</td><td>2.5443e-09</td><td>2</td><td>12.60</td></t<>	(50,2)	LAPM	23	388	3	50	27	27.5866	2.5443e-09	2	12.60
Linob.2.1 LAPR 24 307 9 6 67 19 2441897 1388609 2 10 LURVLID LAPR 23 377 8 67 44 691.997 1388609 2 34.96 LURVLID LAPR 23 208 4 63 39 377.701 7.978610 5 9.40 LURVLID LAPR 32 208 4 63 59 377.701 7.978610 5 24.30 (99.4) LAPR 32 208 4 65 59 6077.623 11855-11 5 24.30 (10.4) LAPR 42 238 44 1038 160 125.873 7 100.61 131 316 62.628.0000 372.171 7 40.29 7.0 100.01 131 316 42.554.41 1038.60 23.554.41 23.556.01 2 7.0 10.57 1.577.14 10.898.10 2 7.0	LUKVLE2	LAPB	26	420	0	380	210	27.5866	9.98/8e-11	5	22.76
CAUSALE LAPE 22 317 8 67 41 094197 41812-09 2 34966 (0.4) LAPC 22 255 6 46 25 377.7011 7.1281-11 5 9.940 (0.4) LAPB 23 288 4 63 39 377.7001 7.1281-11 5 440.01 (0.7,2) LAPB 32 288 4 63 396 637.7000 6.0426e-11 5 440.01 (0.7,2) LAPB 32 1005 1084 3966 637.5000 2.371.615 7 4.50.91 (0.8,4) LAPB 219 1015 1184 1686 645.827.91 1.57.44 2.58.41 1.58.92 1.57.44 1.578.44 2.53.44 1.0138-10 2 9.70 (10.4) LAPB 21 318 0 108 9.4 2.53.44 1.032.40 2.25.44.0 2.20.87 (10.4) LAPB 21 <	(1000.2)	LAPC	24	307	6	63	30	694 1987	3.2490e-11 3.8086e-09	2	19.10
LIKVELE LAPC 28 295 6 46 25 377.7031 8.1838-11 5 9.40 (9,4) LAPR 30 67 371 10 44 377.7031 7.2780-11 5 9.40 (9,4) LAPC 30 371 10 44 39 377.7031 7.2780-11 5 24.30 (99,4) LAPC 32 559 100 103 3060 6337.7000 2.1480-15 7 40.70 (199,49) LAPK 52 419 1010 1131 3166 6263.000 4.406-13 7 4.50.70 (10,4) LAPB 52 241 10 1051 1.5784 1.0580-10 2 7.90 LIKVLE7 LAPC 27 291 1 24 60 -1.5784 0.300-11 5 25.19 LIKVLE7 LAPC 27 238 6 100 10 125.6 5.4081-6.10 5	(1000,2)	LAPB	22	317	8	67	41	694.1987	4.1812e-09	2	34.96
(9.4) LAPB 30 306 9 727031 72280-11 5 9.40 LUKVLE LAPC 30 371 10 44 24 6037.7003 60426-11 5 23.30 UKVLE LAPM 32 558 1009 105 3060 6077.000 1.408-15 7 4.001 UKVLE LAPM 32 293 1005 1084 3066 6073.7000 1.408-15 7 4.020 099.4901 LAPM 52 244 100 1131 3146 62638.2000 4.000-13 7 450.20 (10.4) LAPG 27 291 12 128 177 1.1574 928.00-11 5 12.90 (10.4) LAPM 21 313 2 158 844 -25.9444 6.4122.69 2 12.80 (10.4) LAPM 21 31 4 146 22 12.80 14.81 13.05 5.040.15<	LUKVLE6	LAPC	28	295	6	46	25	377.7031	8.1838e-11	5	10.80
LAPR 23 29 4 63 39 577:01 7222:209 2 17.5 (99.4) LAPC 30 371 10 44 155 360 60077.003 1.2485-11 5 45.30 (99.49) LAPR 52 238 100 1034 3066 65380000 2.21875-11 7 1.229.00 (99.49) LAPR 52 470 1010 1131 3146 62638.000 4.4060e-13 7 450.79 (10.4) LAPR 52 64 128 69 -1.5784 2.554-10 2 7.90 LUKVLE7 LAPR 23 333 6 168 69 -2.54444 6.302-69 2 1.83 (10.4) LAPR 21 333 5 158 84 -2.54444 3.0032-69 2 1.03 (10.4) LAPR 23 311 4 164 86 -3.7627 8.3916-99 2	(9,4)	LAPM	30	306	9	220	121	377.7031	7.2780e-11	5	9.40
LLKVLE LAPM 30 21.0 100 44 50 24.0 0.01248-11 5 24.00 0.9.9 LAPB 52 33 44 105 90 6077523 2055111 7 45511 UKVLE LAPM 52 249 1001 1131 3166 62638000 4.000-13 7 450,70 0994.09) LAPB 54 278 1000 1073 3072 626382379 7.1000-14 7 450,70 (10.4) LAPB 27 291 4 128 60 -1.5784 9.380-10 2 7.90 (10.4) LAPM 23 331 2 158 84 -2.54444 6.4122-69 2 1.30 (10.4) LAPM 23 331 162 86 -1.37627 8.608-11 5 2.150 (10.4) LAPM 23 132 4 466 222 -1.37627 8.608-11 5		LAPB	23	298	4	63	39	377.7031	7.9262e-09	2	17.15
(D7.07) LAPB 12 230 104 105 90 0077,623 10852-11 5 95 95 (099,49) LAPK 52 410 1010 1131 3166 62538000 2.721151 7 1.252.00 (09) LAPK 52 47 1010 1131 3166 62538000 4.400c-13 7 450.32 LUKVLET LAPK 25 264 1 304 163 -1.5784 2.554-10 2 7.90 LUKVLET LAPK 2.3 313 2 158 84 -259444 3102-09 2 12.80 (10.4) LAPB 21 311 4 168 87 -13.7027 8.591-10 2 2.03.7 (10.4) LAPB 21 313 4 168 87 -13.7027 8.591-10 5 2.150 (10.4) LAPB 21 313 4 168 7 13.52 <	LUKVLE6	LAPC	30	371	10	44	24	6037.7000	6.0426e-11	5	24.30
LIEVLE LAPG 29 1005 1064 3086 42538000 42060-13 7 42507 099,499) LAPB 54 278 1009 1073 3072 626382379 7,1000-14 7 450,70 (10,4) LAPB 27 291 1 255 164 1 304 163 -1.5784 9.3800-11 5 25,19 (10,4) LAPB 27 293 4 128 69 -1.5784 9.3800-11 5 25,19 LUEVLE7 LAPC 21 331 2 158 84 -25.9444 6.4123c-69 2 1.18.0 (10,4) LAPC 21 311 4 164 856 -13.7627 8.8761e-9 2 11.50 (10,4) LAPC 27 1444 13 548 257 10.2576 2.5681e-15 5 3.843 LUEVLE7 LAPC 27 1444 13 548 277 </td <td>(99,49)</td> <td>LAPR</td> <td>32</td> <td>294</td> <td>14</td> <td>1059</td> <td>90</td> <td>6037.6523</td> <td>1.0852e=11</td> <td>5</td> <td>25.51</td>	(99,49)	LAPR	32	294	14	1059	90	6037.6523	1.0852e=11	5	25.51
9999 LAPB 52 419 100 1131 3146 623832370 7.1000-14 7 450.39 LUKVLE7 LAPB 2 2 1 1 255 137 -1.5784 1.2854-10 2 9.70 LUKVLE7 LAPB 27 259 4 128 60 -1.5784 0.3806-11 5 25144 (10.4) LAPC 22 338 6 160 95 -253444 6.4122-69 2 1280 (10.4) LAPC 13 31 2 118 84 -253444 6.4122-69 2 1280 (10.4) LAPR 13 31 4 466 22 -137627 8.5311-69 2 1.042 LUKVLE9 LAPC 27 92 10 109 1255 6.5412-11 5 31.69 2 1.642 LUKVLE9 LAPC 27 1423 3.64 2.520 1.0255 4.535	LUKVLE6	LAPC	49	392	1005	1084	3086	62638.0000	2.8721e-15	7	1229.00
LAPR 54 278 1009 1073 3072 62638.279 7.1000-1 7 4502 (10.4) LAPR 27 291 1 255 137 -15784 0.3808-10 2 7.00 (10.4) LAPB 27 289 4 128 69 -15784 0.3806-11 5 25.19 LUKVLET LAPM 23 331 2 158 84 -25.9444 0.038-60 2 12.80 LUKVLET LAPG 21 332 5 158 84 -25.9444 0.038-60 2 1.50 (60.4) LAPG 27 692 7 360 166 1.55 5.8371-60 2 1.50 (10.6) LAPG 27 162 7 466 252 1.3767 4.8485.11 5 21.50 (10.6) LAPG 27 162 7 416 2255 2.5064.11 5 3.15	(999,499)	LAPM	52	419	1010	1131	3146	62638.0000	4.4060e-13	7	450.70
LUKVLE7 LAPK 27 291 1 255 137 -1.5784 1.2854-10 2 9,70 LUKVLE7 LAPB 27 259 4 128 69 -1.5784 0.3806-11 2 128 (100,4) LAPC 22 338 6 169 95 -253444 6.4122-69 2 1280 (100,4) LAPB 21 331 2 158 84 -25.94444 10.038-69 2 12.80 (100,4) LAPB 21 32 156 84 -25.94444 10.038-69 2 9.02 LUKVLE9 LAPB 23 3 166 85 -13.7627 8.401-69 2 9.02 100,6 LAPM 27 1424 13 548 297 10.2355 4.2081-61 5 35.20 (100,6) LAPM 27 1407 10 514 22.5564 7.3856-11 5 40.00		LAPB	54	278	1009	1073	3072	62638.2379	7.1000e-14	7	450.29
(10.4) LAPB 25 264 1 304 163 -1.5784 2.2534-0 2 730 LUKVLE7 LAPC 22 338 6 169 95 -2.53444 1.0386-07 2 14.80 (100,4) LAPM 21 332 5 158 84 -2.53444 1.0038-07 2 2.087 (100,4) LAPC 21 311 4 164 87 -1.37627 8.5971-697 2 9.09 (10.6) LAPB 22 312 4 466 2.525 5.4083-11 5 2.192 LUKVLE9 LAPB 27 692 7 340 196 1.2556 5.4083-11 5 3.83 UKVLP LAPB 27 1027 7 340 196 1.2556 5.4085-11 5 3.70 UKVLP LAPM 27 1027 7 46 2.5560 3.8124-11 5 9.17 <t< td=""><td>LUKVLE7</td><td>LAPC</td><td>27</td><td>291</td><td>1</td><td>255</td><td>137</td><td>-1.5784</td><td>1.0880e-10</td><td>2</td><td>9.70</td></t<>	LUKVLE7	LAPC	27	291	1	255	137	-1.5784	1.0880e-10	2	9.70
LIKVLEP LAPP 2/2 2/3 3/4 6 1/6 9/4 -1/2/44 9/2/44 9/2/45 2/1 3/1 2 1/3 (100,4) LAPC 21 3/3 2 1/3 8 4 -25/3444 1/0/25 2 1/2 1/2 LIKVLE7 LAPC 21 3/3 1/2 4 1/6 8 -7/3/37 8.69716-00 2 1/2 1/2 LIKVLE7 LAPC 27 6/92 7/3 3/0 1/6 2/2 5/0/3 1/6 2 1/6 2 1/6 2/2 1/6 2 1/6 2 1/6 1/6 2/2 1/2 1/6 2 1/6 3/3 3/3 1/6 3/3 1/2 1/4 3/3 1/4 1/2 1/2 1/4 1/2 1/2 1/4 1/2 1/2 1/4 1/2 1/2 1/4 1/2 1/2 1/2 1/2 1/2 1/2	(10,4)	LAPM	25	264	1	304	163	-1.5784	2.2554e-10	2	7.90
LUKLID LAPR 23 33 2 158 84 -25 5444 00038-00 2 1 180 (100,4) LAPR 21 321 52 158 84 -25 5444 30635-00 2 1150 (50,4) LAPM 19 211 311 4 164 87 -137677 8.571(-00) 2 11.50 (100,6) LAPM 22 312 4 466 252 -137677 8.571(-00) 2 16.42 (100,6) LAPC 27 799 9 100 109 1.2526 6.813(-11 5 5.250 38.33 1.5 3.134 1.5 5.250 3.134(-11 5 5.220 10.355-11 5 9.220 1.0285 4.218(-11 5 9.220 1.0285 4.218(-11 5 9.220 1.02864 5.721(-11 5 9.194 1.1247 1.0280 1.23 1.1312 4.165 4.99 1.3 1.23 3.132	LUKVLE7	LAPB	27	259	4	128	05	-1.5/84	9.3809e-11	5	25.19
LAPE 21 352 5 158 84 -25 0444 30673-09 2 2087 ULWUED LAPC 12 31 4 164 87 -13767 8.6911-09 2 900 (50.4) LAPB 22 312 4 466 252 -137627 8.6911-09 2 900 (10.6) LAPB 28 719 9 190 109 12256 5.4038-11 5 21.50 (10.6) LAPH 28 876 8 700 385 12.256 5.4038-11 5 35.3116-09 2 5 2.50 (10.6) LAPH 29 1065 7 416 225 10.2857 7.2888-11 5 37.10 5 3.70 ULWUE9 LAPC 27 1403 12 418 259 5.264 3.8134-11 5 39.80 (50.6) 1.813 4.41576-09 2 7.10 (50.6) LAPH	(100.4)	LAPM	23	331	2	158	84	-25.9444	1.0038e-09	2	12.80
	(,-)	LAPB	21	352	5	158	84	-25.9444	3.0623e-09	2	20.87
(50.4) LAPB 199 281 3 162 86 -1.3.7627 8.681e-09 2 9.90 LUKVLE9 LAPC 27 692 7 340 196 1.2826 5.481e-01 5 21.50 LUKVLE9 LAPM 28 836 8 700 385 1.2826 2.680e-11 5 38.3 (10.6) LAPM 27 1424 13 548 277 10.2857 4.288e-11 5 37.10 (10.6) LAPM 29 1065 7 416 22 5.2650 38.13e-11 5 39.80 (50.6) LAPM 27 1103 7 26 145 5.2650 38.13e-11 5 39.80 (10.8) LAPM 22 1372 8 3349 1843 3.1152 3.228e-11 5 4.4 (10.8) LAPM 28 189 12 25 144 3.1152 3.228e-11 <	LUKVLE7	LAPC	21	311	4	164	87	-13.7627	8.5971e-09	2	11.50
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(50,4)	LAPM	19	281	3	162	86	-13.7627	8.6081e-09	2	9.90
LUKVLE9 LAPM 27 692 7 340 196 1.2226 5.488-11 5 21.50 LUKVLE9 LAPE 28 836 8 700 385 1.2326 2.5608-11 5 38.33 (100,0) LAPE 27 1424 13 548 297 102.857 7.2388-11 5 37.10 (100,0) LAPE 27 1103 7 256 145 5.2650 38.12e-11 5 39.80 (50,6) LAPM 27 150 12 418 22 5.2654 7.8305-11 5 4.40 (10,8) LAPM 28 147 10 257 144 3.1152 3.2280-11 5 4.40 (10,8) LAPM 28 189 12 255 140 17.2467 5.6457e-11 5 7.70 LUKVLE10 LAPC 24 132 5 3.40 17.2467 5.6457e-11 5		LAPB	22	312	4	466	252	-13.7627	8.3511e-09	2	16.42
(10.0) LAPB 2.3 1/9 9 1/9 </td <td>LUKVLE9</td> <td>LAPC</td> <td>27</td> <td>692</td> <td>7</td> <td>340</td> <td>196</td> <td>1.2526</td> <td>5.4083e-11</td> <td>5</td> <td>21.90</td>	LUKVLE9	LAPC	27	692	7	340	196	1.2526	5.4083e-11	5	21.90
	(10,6)	LAPM	28	836	8	700	385	1.2520	0.8512e-11 2.5608e-11	5	21.50
	LUKVLE9	LAPC	2.7	1424	13	548	297	10.2855	4.2181e-11	5	52.20
LAPB 27 1407 10 514 283 10.2864 5.7211e-11 5 91.94 (50,6) LAPM 27 1103 7 256 145 5.2650 3.8124-11 5 3.980 (50,6) LAPB 27 1250 12 418 229 5.2654 7.8305e-11 5 7.47 LUKVLEI0 LAPC 21 151 6 349 1844 3.1152 6.6179-09 2 5.43 (10.8) LAPB 20 99 9 233 123 3.1152 6.6179-09 2 7.10 (10.KVLEI0 LAPC 24 145 4 476 257 17.2468 6.1679-09 2 7.10 (10.WLEI0 LAPC 29 304 6 37.4 204 34.2425 8.2357e-11 5 1.14 (10.08) LAPM 30 180 82 213 34.94244 6.5142-09 2.13.0 <tr< td=""><td>(100,6)</td><td>LAPM</td><td>29</td><td>1065</td><td>7</td><td>416</td><td>225</td><td>10.2857</td><td>7.2888e-11</td><td>5</td><td>37.10</td></tr<>	(100,6)	LAPM	29	1065	7	416	225	10.2857	7.2888e-11	5	37.10
LUKVLE9 LAPC 27 1103 7 256 145 5.2654 3.8124-11 5 39.80 (50,6) LAPB 25 1372 8 3349 1843 5.2653 7.8305-11 5 40.40 (10.8) LAPB 21 151 6 349 1844 3.1152 41.6757-09 2 5.4 (10.8) LAPB 20 99 9 233 123 3.1152 6.61679-09 2 7.10 (50.48) LAPM 28 189 12 255 140 17.2467 8.3827-11 5 11.8 LUKVLE10 LAPC 29 173 11 257 114 17.2467 8.3857-11 5 11.8 LUKVLE10 LAPC 29 304 6 374 204 34.9244 26.562-11 5 9.00 LUKVLE10 LAPB 24 142 6 382 213 34.9244 26.562-11		LAPB	27	1407	10	514	283	10.2864	5.7211e-11	5	91.94
(50.6) LAPM 27 12 112 418 229 5.2654 7.8305-11 5 40.40 LUKVLEI0 LAPC 21 151 6 349 1843 5.2653 9.0057-11 5 72.47 LUKVLEI0 LAPM 28 147 10 257 1144 3.1152 3.2280-11 5 4.50 LUKVLEI0 LAPM 28 189 12 257 172.468 6.6514-09 2 7.10 (0.0,8) LAPM 28 189 12 257 141 17.2467 5.6457e-11 5 1.188 LUKVLEI0 LAPB 29 173 11 257 141 17.2467 8.2357e-11 5 1.188 (100.98) LAPM 30 180 8 206 111 3.49244 6.5142-09 2 2.13 3.49244 5.15557 12.04 1.309244 6.5142-09 2 1.30 1.155 12.04 1.139244<	LUKVLE9	LAPC	27	1103	7	256	145	5.2650	3.8124e-11	5	39.80
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(50,6)	LAPM	27	1250	12	418	229	5.2654	7.8305e-11	5	40.40
	LUKVI E10	LAPB	25	15/2	8	3349	1845	3.2033	9.0057e-11	2	5.40
LAPB 20 99 9 233 123 3.1152 6.6514-09 2 5.74 LUKVLEI0 LAPC 24 155 4 476 257 17.2468 6.6514-09 2 5.74 LUKVLEI0 LAPC 24 155 4 476 257 141 17.2467 5.6457e-11 5 1.50 LUKVLEI0 LAPB 29 173 11 257 141 17.2467 8.3255e-11 5 9.00 LUKVLEI1 LAPM 30 180 8 206 111 34.9244 6.5142-09 2 21.30 LUKVLEI1 LAPC 29 251 7 294 158 0.0005 4.1139e-11 5 1.595 LUKVLEI1 LAPC 31 301 10 504 281 0.0007 3.6638-09 2 19.86 LUKVLEI1 LAPC 31 327 7 719 402 0.00012 7.3748-11	(10.8)	LAPM	28	147	10	257	144	3.1152	3.2280e-11	5	4.50
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(10,0)	LAPB	20	99	9	233	123	3.1152	6.6514e-09	2	5.74
	LUKVLE10	LAPC	24	155	4	476	257	17.2468	6.1679e-09	2	7.10
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(50,48)	LAPM	28	189	12	255	140	17.2467	5.6457e-11	5	7.50
		LAPB	29	173	11	257	141	17.2467	8.3827e-11	5	11.88
	(100.08)	LAPC	29	304	0	3/4	204	34.9245	8.2355e-11	5	21.40
	(100,98)	LAPB	24	142	6	382	213	34.9244	6.5142e-09	2	21.30
	LUKVLE11	LAPC	29	252	8	42	23	0.0002	6.2489e-11	5	10.50
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(49,30)	LAPM	29	251	7	294	158	0.0005	9.2138e-11	5	8.20
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAPB	29	281	8	247	136	0.0005	4.1139e-11	5	15.95
(98,64) LAPM 29 302 8 171 95 0.0010 6.89/2e-11 5 1.290 LAPB 23 257 7 719 402 0.0007 3.6683-00 2 19.86 LUKVLE11 LAPC 31 325 8 270 146 0.0012 7.2189-11 5 974.60 (998,664) LAPB 30 332 7 434 240 0.0002 9.5707e-11 5 484.60 (49.30) LAPM 28 4858 9 4610 2339 319.9856 4.7674e-11 5 190.70 LKVLE13 LAPC 29 5698 9 859 469 319.9856 4.1940e-11 5 336.25 LUKVLE13 LAPC 32 3586 15 688 373 776.8122 4.8749e-11 5 245.20 LUKVLE13 LAPM 32 3514 9 293 162 729.2834 6.5300e-11	LUKVLE11	LAPC	31	301	10	504	281	0.0008	4.0865e-11	5	14.90
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(98,64)	LAPM	29	302	8	1/1	95	0.0010	6.89/2e-11	5	12.90
	LUKVLE11	LAPC	31	325	8	270	146	0.0007	7.2189e=11	5	974.60
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(998,664)	LAPM	30	352	8	239	132	0.0005	4.6918e-11	5	484.60
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		LAPB	30	332	7	434	240	0.0002	9.5707e-11	5	464.72
	LUKVLE13	LAPC	29	5434	5	569	312	319.9856	4.7674e-11	5	190.70
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	(49,30)	LAPM	28	4858	9	4610	2539	319.9856	9.2916e-11	5	151.90
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	LUKVI E12	LAPB	29	2098	9	839 589	409	319.9850	4.1940e-11 4.8740a-11	5	223 20
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(98.64)	LAPC	32	3586	15	688	376	729,2835	4.67490-11 7.5921e-11	5	148.10
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	(20,01)	LAPB	32	3514	9	293	162	729.2834	6.5300e-11	5	245.20
	LUKVLE13	LAPC	7	1147	3	18050	9550	5822.1000	0.2974	8	3860.20
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(998,664)	LAPM	7	2523	4	59801	31673	6094.6000	8.3979	8	3693.20
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		LAPB	7	2589	1	63960	33658	6094.5698	8.3980e+00	8	3880.60
	LUKVLE15 (57.42)	LAPC	29	472	5	124	67	0.0071	9.3265e-11	5	18.80
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	(37,42)	LAPM	30	415	8	537	299	0.0050	8.6258e-11	5	25.50
(97,72) LAPM 30 785 6 260 141 0.0738 4.6129c-11 5 32.80 LLXPE 31 838 7 933 515 0.0738 4.6129c-11 5 32.80 LUKVLE15 LAPE 31 838 7 933 515 0.0738 4.6129c-11 5 57.87 LUKVLE15 LAPE 33 839 7 65 36 0.4771 1.6966c-11 5 51.11.44.90 (997,47) LAPB 31 853 4 248 138 0.5708 5.4842c-11 5 1649.30 LUKVLE16 LAPB 31 853 4 248 138 0.5708 5.4842c-11 5 19.20 (57,42) LAPM 29 562 7 266 147 0.0001 3.7020c-11 5 19.20 LVKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020c-11 5	LUKVLE15	LAPC	32	520	4	369	206	0.0073	5.6284e-11	5	26.70
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	(97,72)	LAPM	30	785	6	260	141	0.0738	4.6129e-11	5	32.80
LUKVLE15 LAPC 33 839 7 65 36 0.4771 1.6966e-11 5 331.10 (997,747) LAPM 32 873 10 514 289 0.5719 8.0766e-11 5 1444.90 LAPM 32 873 10 514 289 0.5708 5.4842e-11 5 1649.30 LUKVLE16 LAPC 30 559 9 707 393 0.0004 3.2864e-11 5 1649.30 LUKVLE16 LAPM 29 562 7 266 147 0.0003 8.8242e-11 5 19.20 LAPB 28 528 9 526 292 0.0001 7.9811e-11 5 40.63 LUKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020e-11 5 26.20 (97,72) LAPM 30 519 7 779 432 0.0008 4.9252e-11 5 19.60<		LAPB	31	838	7	933	515	0.0738	4.5694e-11	5	57.87
(997,747) LAPM 32 873 10 514 289 0.5719 8.0766e-11 5 1444.90 LAPB 31 853 4 248 138 0.5708 5.4842e-11 5 1649.30 LUKVLE16 LAPC 30 559 9 707 393 0.0004 3.2864e-11 5 1649.30 (57,42) LAPM 29 562 7 266 147 0.0001 7.9811e-11 5 19.20 LAFB 28 528 9 526 292 0.0001 7.9811e-11 5 40.63 LUKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020e-11 5 26.20 (97,72) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 19.60 LAPB 29 494 8 465 251 0.0008 2.528-11 5 31.45	LUKVLE15	LAPC	33	839	7	65	36	0.4771	1.6966e-11	5	3311.10
LAPB 51 853 4 248 138 0.5708 5.4842e-11 5 1649.30 LUKVLE16 LAPC 30 559 9 707 393 0.0004 3.2864e-11 5 128.80 (57,42) LAPM 29 562 7 266 147 0.0001 3.8264e-11 5 19.20 LUKVLE16 LAPB 28 528 9 526 292 0.0001 7.9811e-11 5 40.63 LUKVLE16 LAPB 28 587 3 822 455 0.0010 3.7020e-11 5 26.20 (97.72) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 19.60 LAPB 29 494 8 465 251 0.0008 2.528s-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 33956e-11 5 79.30	(997,747)	LAPM	32	873	10	514	289	0.5719	8.0766e-11	5	1444.90
LUKVLE16 LAPC 30 50 52 707 393 0.0003 8.8242e-11 5 19.20 L(57,42) LAPB 28 552 7 266 147 0.0003 8.8242e-11 5 19.20 LAPB 28 528 9 526 292 0.0001 7.9811e-11 5 40.63 LUKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020e-11 5 26.20 (97,7) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 19.60 LAPM 29 494 8 465 251 0.0008 2.5283e-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30	LUKVI E14	LAPB	31	833	4	248	1.58	0.5708	3.4842e-11	5	1049.30
LAPB 28 528 9 526 292 0.001 7.9811e-11 5 40.63 LUKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020e-11 5 26.202 (97,72) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 26.202 LAPB 29 494 8 465 251 0.0008 2.5283e-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30	(57.42)	LAPC	20	562	7	266	393 147	0.0004	3.20040-11 8.8242e-11	5	19.20
LUKVLE16 LAPC 31 587 3 822 455 0.0010 3.7020e-11 5 26.20 (97.72) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 19.60 LAPB 2.9 494 8 465 251 0.0008 2.5283e-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30	(27,72)	LAPB	28	528	9	526	292	0.0001	7.9811e-11	5	40.63
(97,72) LAPM 30 519 7 779 432 0.0008 4.9525e-11 5 19.60 LAPB 29 494 8 465 251 0.0008 2.5283e-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30	LUKVLE16	LAPC	31	587	3	822	455	0.0010	3.7020e-11	5	26.20
LAPB 29 494 8 465 251 0.0008 2.5283e-11 5 31.45 LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30	(97,72)	LAPM	30	519	7	779	432	0.0008	4.9525e-11	5	19.60
LUKVLE16 LAPC 31 534 9 864 470 0.0028 3.3956e-11 5 79.30		LAPB	29	494	8	465	251	0.0008	2.5283e-11	5	31.45
	LUKVLE16	LAPC	31	534	9	864	470	0.0028	3.3956e-11	5	79.30

D. LL	Mala			1	# C	# 2 7.0	((-)	Table 1 – continu	ed from p	revious page
(n, m)	Method	it.ext	it.int	search	#L	#VL	f(x)	h(x)	exit	(seconds)
(197,147)	LAPM	33	485	7	363	203	0.0026	7.2338e-11	5	28.70
1000	LAPB	30	434	8	354	194	0.0026	6.0581e-11	5	38.58
(2.1)	LAPC	17	22	6	64 77	34 46	-1.0000	7.6983e-09 1.7306e-09	2	0.80
(2,1)	LAPB	17	21	6	77	46	-1.0000	1.7306e-09	2	1.40
METHANB8	LAPC	2	10002	0	343802	184159	0.0000	0.2609	6	357.40
(31,31)	LAPM	2	10002	0	333740	178091	0.0000	0.2516	6	317.30
METHANI 8	LAPB	2	10002	0	334424	1/8/8/	0.0000	2.4965e-01	6	357.90
(31,31)	LAPM	2	10002	0	339989	181656	0.0000	1.0481	6	318.40
	LAPB	2	10002	0	339795	181865	0.0000	1.4344e+00	6	608.77
MSS1 (00.73)	LAPC	30	208	7	50	27	-9.0000	9.9696e-10 5.3058a.07	2	10.00
(90,75)	LAPB	27	10590	13	166167	90269	-16.0000	1.4773e-07	6	654.48
MWRIGHT	LAPC	23	251	8	59	37	1.2884	3.3691e-09	2	8.80
(5,3)	LAPM	23	249	4	168	89	1.2884	6.8137e-09	2	8.20
OPTHPDM2	LAPB	23	240	4	2/8	159	1.2884	1.1419e-09	2	14.68
(53,25)	LAPM	21	5804	6	4636	2695	1.7974	8.6810e-11	5	138.00
	LAPB	21	6342	4	6021	3494	1.7974	2.1055e-11	5	281.29
ORTHRDM2	LAPC	23	5236	5	2945	1725	3.9805	6.9584e-11	5	155.30
(102,50)	LAPM	24	4131	5	010	484	3.9803	7.2705e-11 2.7790e-11	5	308.45
ORTHRDM2	LAPC	24	4254	5	1542	911	7.7757	2.4915e-11	5	461.60
(203,100)	LAPM	16	6947	3	94	55	7.7757	9.8219e-09	2	368.30
operation	LAPB	10	10218	4	17504	9573	7.7758	5.1309e-07	6	664.66
(133.64)	LAPC	5	13009		294995	158533	9.4119	2.7202	6	520.70
(155,64)	LAPB	4	11481	1	299039	161240	6.9835	2.7794e+00	6	678.23
ORTHREGB	LAPC	26	877	8	1314	743	0.0001	4.7513e-11	5	24.00
(27,6)	LAPM	25	1023	4	614	354	0.0001	3.2900e-11	5	27.00
ORTHREGC	LAPB	30	415	7	1060	86	0.0001	7.4605e=11	5	00.35
(25,10)	LAPM	27	508	8	118	64	0.3993	6.3794e-11	5	19.80
	LAPB	27	390	10	163	91	0.3992	5.9452e-11	5	20.12
ORTHREGC	LAPC	30	1195	5	2205	1223	1.9768	2.4198e-11	5	51.00
(105,50)	LAPM	29	1057	7	1043	632	1.9767	9.8628e-11	5	40.90
ORTHREGC	LAPC	30	3078	3	1656	929	9.5822	9.1101e-11	5	1843.20
(505,250)	LAPM	29	3134	6	690	392	9.5823	9.3743e-11	5	639.80
OPTUPECC	LAPB	28	2773	7	2494	1405	9.5822	9.3231e-11	5	1535.40
(1005.500)	LAPM	13	3553	7	13403	7217	18.7911	2.0683e-05	8	3688.30
	LAPB	5	3591	4	81769	43425	18.7724	4.5752e-03	8	3966.40
ORTHREGD	LAPC	23	7691	3	1357	787	15.5906	7.8543e-11	5	267.00
(103,50)	LAPM	25	3447	3	260	96	15.5904	9.8530e-11 9.6739e-11	5	99.40
ORTHRGDM	LAPC	20	1547	6	338	192	3.2886	4.6668e-11	5	44.80
(23,10)	LAPM	20	2493	2	1425	812	3.2886	6.2864e-11	5	62.70
OPTUDODM	LAPB	23	2573	2	1318	772	3.2886	3.3585e-11	5	108.20
(103.50)	LAPC	23	7748	6	884	510	15.3702	9.2850e-11	5	217.00
、	LAPB	26	9183	5	1171	688	15.3704	4.8554e-11	5	379.92
ORTHRGDM	LAPC	24	6235	3	1116	671	23.3348	3.3362e-11	5	357.50
(155,76)	LAPM	23	9974		673	394 608	23.3348	1.3969e-10 4.2853e-11	2	345.20
POWELLBS	LAPC	14	10080	1	89000	53606	0.0000	0.0015	6	91.10
(2,2)	LAPM	14	10378	7	79749	49520	0.0000	0.0011	6	83.50
DOWERLAG	LAPB	15	10996	7	68170	42838	0.0000	8.1382e-04	6	137.36
POWELLSQ (2.2)	LAPC	6	12496		190107	104254	0.0000	2.0115e-05 7.3204e-06	0	210.30
(2,2)	LAPB	6	34	ŏ	1	1	0.0000	7.3204e-06	1	1.64
RECIPE	LAPC	23	98	3	415	228	0.0000	7.7438e-08	1	3.40
(3,3)	LAPM	25	91	10	250	137	0.0000	3.7290e-09	2	3.10
RSNBRNE	LAPB	2.5	238	3	851	41	0.0000	0.97.50e-09 1.6742e-08	2	4.80
(2,2)	LAPM	15	126	6	350	200	0.0000	2.6042e-06	1	3.90
	LAPB	15	153	8	1196	644	0.0000	1.1082e-07	1	8.83
S316-322	LAPC	32	99	3	129	68	334.3146	5.5905e-10	2	3.70
(2,1)	LAPM	25	111		129	68	334.3146	2.6068e-09		7.12
SINVALNE	LAPC	14	214	3	360	199	0.0000	2.2052e-06	1	7.30
(2,2)	LAPM	22	319	6	1607	586	0.0000	3.8510e-09	2	8.60
	LAPB	21	286	2	368	195	0.0000	1.9198e-09	2	15.94

In order to simplify the analysis of the results, we have built smaller tables with specific information. Tables 2, 3 and 4 refer to the number of times that each methodology, LAPC, LAPB and LAPM, performed better, taking into account the number of outer iterations of the algorithm (*it.ext*), the number of iterations of the inner algorithm (*it.int*), the number of augmented Lagrangian function evaluations (*func*) and its

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gradient (*grad*), the number of searches (*search*) and the computational time (*time*). The values related to the tie indicate the amount of problems of which performance coincided.

We consider that the algorithm failed to obtain the solution of the problem in cases in which *exit* assumed values 6, 7 or 8. Under these conditions, the methods LAPC, LAPM and PAPB did not obtain the solution of 19, 20 and 21 problems, respectively. In total, 24 problems were not resolved for at least one of the methods and, between these, 17 were not resolved by any of the three methodologies.

The fact that we allow variations in the accuracy of the stopping criterion, related to feasibility and optimality, can anyway, contribute for the determination of different stationary points and that demand less computational effort of the methods. Thus, to ensure the realization of a proper comparison, we consider only the problems in which methods determined the same solution, that is, we discarded of our analysis the problems that

$$|f(\bar{x}) - f_{min}(\bar{x})| > 10^{-3} \tag{24}$$

where, $f_{min}(\bar{x})$ is the lower value functional in \bar{x} for each problem presented in the Table 1.

Thus, besides the 24 ones, 13 more problems were also discarded based on the condition (24), therefore we analyzed the results of 97 problems.

The following is the comparisons between methods: LAPC and LAPM (Table 2), LAPC and LAPB (Table 3) and finally, LAPM and LAPB (Table 4).

Table 2: Comparison between the algorithms LAPC and LAPM

Method	it.ext	it.int	func	grad	search	time
LAPC	28	42	45	46	45	23
LAPM	53	55	49	48	42	71
tie	16	0	3	3	10	3

The results presented in Table 2, indicate that the LAPM method obtained a better performance than the LAPC method compared with respect to *it.ext, it.int, func, grad* and *time*, in 65.43%, 56.70%, 52.13%, 51.06% and 75.53% of cases, respectively. Although the LAPC method has shown best results for the number of searches, it did not influenced the occurrence of best performances for the remaining criteria. On the contrary, against other criteria, especially with respect to computational time, the results on the LAPM were better than the results achieved by the LAPC method. This fact is a direct consequence of the realization of a smaller amount of inner iterations of the algorithm by the LAPM method, because the greatest computational effort of augmented Lagrangian methods is associated with the determination of the solution of unconstrained subproblems (minimizing the augmented Lagrangian function) and in the external algorithm, it is performed only the updating of penalty parameter and the Lagrange multiplier.

Regarding the number of iterations performed, for both external and internal algorithms, the results of Table 3 show that the LAPB methodology was more efficient

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Method	it.ext	it.int	func	grad	search	time
LAPC	31	45	54	53	45	66
LAPB	49	51	40	41	42	31
tie	17	1	3	3	10	0

Table 3: Comparison between the algorithms LAPC and LAPB

in 61.25% and 53.13% of the cases, respectively, when compared to LAPC method. On the other hand, the performance of the LAPC methodology was better than LAPB concerning to a number of function evaluations, number of gradient evaluations, number of search and to computational time in 57.45%, 56.38%, 51.72% and 68.04% of the 97 problems analyzed, respectively.

Table 4: Comparison between the algorithms LAPM and LAPB

Method	it.ext	it.int	func	grad	search	time
LAPM	29	41	51	49	29	79
LAPB	35	38	25	27	33	18
tie	33	18	21	21	35	0

The results presented in Table 4, indicate a similar performance to LAPM and LAPB concerning *it.ext* and *it.int*. Note that, the LAPB methodology outperformed the LAPM method, on the number of external iterations in 6 of 97 problems and the number of searches in only 4 problems. On the other hand, we observed the significant performance of the LAPM method concerning to computational time, which took less time to be executed than LAPB method in 81.44% of problems analyzed. In fact, this result is caused by the lower amount of function and gradient evaluations by the LAPM method, which contributed strongly in reducing the computational time to solve the problem.

In general terms, the augmented Lagrangian method with the use of augmented Lagrangian function (4), compared to the other two methods, was more efficient in the number of inner iterations, number of function and gradient evaluations, contributing in some way, in reducing the computational effort, significantly improving the computational time spent executing the algorithm.

5 Conclusion

In this paper we have introduced two new augmented Lagrangian methods applied to resolution of problems with equality constraints. We have extended the results presented by Gonzaga and Matioli in [17] and by Tseng and Bertsekas in [27] for problems with inequality constraints. Our motivation was that the penalties discussed in this paper showed good performance as much of theoretical point of view as computational. Furthermore, the results shown by Martinez, Castilho and Birgin [3] indicating that the classical augmented Lagrangian methods have superiority to modern methods, encouraged us to introduce new augmented Lagrangian methods with features similar to the classics, but with properties of modern methods.

For the methods proposed in this paper, we proved that under second order sufficient conditions the augmented Lagrangian function has local minimizer.

The two new methods presented, constitute alternative methodologies for the resolution of minimization problems with equality constrains. Numerical results indicated that the LAPM is a competitive and promising method when compared with the other two methodologies, because this method presented best results, mainly, concernig to computational time, number of inner iterations, number of function and gradient evaluations.

Another important aspect to emphasize is on the penalty parameter. We show that for a certain choice of the proposed methodologies coincide with the classical method proposed by Hestenes [11] and Powell [22]. From the practical point of view, it was evident that a good choice of this parameter will increase the speed of convergence. Inspecting the geometry and some results already achieved on these penalties, we have obtained good numerical results compared to the conventional method. As a future research we suggest the investigation of a better choice of the penalty parameter, including theoretical studies that support unproven statements in this paper and in other ones.

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