A posteriori error analysis of Galerkin finite element method for the transient Stokes equations☆

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Abstract

In this work, a posteriori error estimates of the finite element method are studied for the transient Stokes equations. With the help of an appropriate Stokes reconstruction, a posteriori error estimates of the velocity and pressure with optimal order of convergence are provided in $L^\infty(L^2)$, $L^\infty(H^1)$ and $L^2(L^2)$ norms for the semi-discrete scheme. Moreover, a fully discrete formulation is proposed. For the full discrete formulation a posteriori error estimates are derived based on the backward Euler scheme. The new results improve and extend a posteriori error estimates for the time-dependent Stokes equations. Finally, some numerical examples are provided to verify the efficiency of the established error estimators.

Keywords: Posteriori error estimates; Transient Stokes equations; Stokes reconstruction; Backward Euler scheme

2000 MSC: 65N15, 65N30, 76D07.

☆This work was supported by CAPES and CNPq, Brazil, and the first author was partly supported by the Doctor Fund of Henan Polytechnic University (B2012-098).

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1. Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with a sufficiently smooth boundary. We consider the following incompressible transient Stokes equations

$$\begin{cases} u_t - \Delta u + \nabla p = f, & \text{in } \Omega \times (0,T], \\ \text{div } u = 0 & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times (0,T], \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t))^T$ is the velocity, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, u_0 the initial velocity, and $T > 0$ a finite time.

A posteriori error estimators have been well studied and used to derive adaptive mesh refinement for solving the time-dependent equations [1, 2, 3, 4]. From the theoretical point of view, some novel analytical techniques need to be introduced to obtain the optimal error estimates. While from the practical use point of view, one needs to reduce the computational time in order to obtain a satisfactory accuracy in the numerical simulations. Although the theory of a posteriori error estimates of the finite element method for elliptic problems is well-developed [5, 27], the theory for time-dependent problems is less developed. Only a few results have been published till now, such as [12, 18, 23, 28, 29] and the reference therein. In recent works [11, 20, 21], Makridakis and his co-workers have introduced the elliptic reconstruction to treat the linear parabolic equation in both semi-discrete and fully discrete schemes. By using the energy techniques and dual arguments, they have obtained optimal a posteriori error estimates. The advantages of introducing an elliptic reconstruction lie in (i) Compared with [12, 13, 14], no-refinement assumption about the mesh can be relaxed; (ii) The well-developed theoretical results of a posteriori error estimates for elliptic problem can be used to obtain the optimal order posteriori error estimates for the parabolic problems; (iii) The process of analysis becomes quite clear and straightforward. Based on these superiorities, the elliptic reconstruction technique has been further developed for both Galerkin method [16] and mixed finite element method [22].

For a posteriori error estimates of the finite element method for the unsteady Stokes problem, Nicaise and Soualem have proposed the semi-discrete time/space and completely discrete space-time schemes with the upper and lower bounds for the spatial estimator in [24, 25], respectively based on Crouzeix-Raviart's spaces. A fully discretized scheme was analyzed with the errors of velocity in $L^2(L^2)$ and $L^\infty(L^2)$ norms in [6, 8]. Using an appropriate Stokes reconstruction, Karakatsani and Makridakis have provided residual-based error estimates in $L^2(H^1)$, $L^\infty(H^1)$ and $L^\infty(L^2)$ norms for the velocity based on Crouzeix-Raviart's elements and divergence-free condition in [19]. By establishing the equivalence between errors and residual and using a suitable decomposition of the residual into spatial and temporal contributions, Verf*ü*rth [30] has obtained a posteriori error bounds for velocity in $L^2(H^1)$ and $L^\infty(L^2)$ norms. In this article, we shall construct an appropriate Stokes reconstruction depending on the residuals and derive a posteriori error estimates in $L^\infty(L^2)$, $L^\infty(H^1)$ and $L^2(L^2)$ norms with optimal order of convergence for the velocity and pressure. Compared with well-known results, the novel ingredients of this work lie in:

(1) The analytical techniques are different. Here, a Stokes reconstruction is constructed to solve the time-dependent Stokes problem. In this way, the analysis is quite straightforward and the heart of the matter is succinct, in contrast with the involved approaches mentioned above. Although the Stokes reconstruction technique was adopted in [19], our operator is quite different from that ones, not only in expressions but also in properties (Please see Definitions 3.1 and 4.1 for details).

(2) A general analysis of a posteriori error estimates of the finite element method for the time-dependent Stokes problem is established, rather than restricting on some special finite element spaces. (divergence-free space is used in [19])

(3) A posteriori error estimates of pressure in $L^2(L^2)$ and $L^\infty(L^2)$ norms are also provided based on the continuous inf-sup condition and dual arguments.

This article is organized as follows. In Section 2, we formulate the finite element method and recall some basic lemmas. In Section 3, we present a posteriori error estimates of the finite element method for space semi-discrete formulation. In Section 4, we derive a posteriori error estimates for the velocity and pressure in fully discrete formulation. Finally, some numerical experiments are provided in Section 5 to verify the performances of established error estimators.

2. Preliminaries

In this section, we formulate the finite element method for the transient Stokes equations (1.1) and recall some basic results.

For the mathematical setting of problem (1.1), we denote

$$X = H_0^1(\Omega)^2, \;\; Y = L^2(\Omega)^2, \;\; D(A) = H^2(\Omega)^2 \cap X,$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

In this work, standard Sobolev spaces are used. The spaces $L^2(\Omega)^m$ $(m = 1, 2, 4)$ are endowed with the standard L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\|\cdot\|_0$. The spaces $H_0^1(\Omega)$ and X are equipped with the norm $\|\cdot\|_1^2$ and seminorm $|\cdot|_1$, respectively. Throughout this paper, the letter $C > 0$ denotes a generic constant, independent on mesh parameter and time step, and may be different at different occurrences.

Furthermore, we assume that the data u_0 and f satisfy the assumption:
(**A1**): The initial velocity $u_0 \in D(A)$ with div $u_0 = 0$ and $f, f_t, f_{tt} \in L^2(0, T; Y)$ are assumed to satisfy

$$\|u_0\|_2 + (\int_0^T (\|f\|_0^2 + \|f_t\|_0^2 + \|f_{tt}\|_0^2)dt)^{\frac{1}{2}} \leq C.$$

The continuous bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $X \times X$ and $X \times M$ are, respectively, defined by

$$a(u, v) = (\nabla u, \nabla v), \quad d(v, q) = -(\nabla q, v) = (q, \mathrm{div} v).$$

Obviously, the bilinear $a(\cdot, \cdot)$ is continuous and coercive on $X \times X$. Also the bilinear $d(\cdot, \cdot)$ is continuous on $X \times M$ and satisfies the well-known inf-sup condition [15, 26]: There exists a constant $\beta > 0$, for all $q \in M$ such that

$$\beta\|q\|_0 \leq \sup_{0 \neq v \in X} \frac{|d(v, q)|}{\|\nabla v\|_0}. \tag{2.1}$$

With above notations, the variational formulation of problem (1.1) reads as: For $\forall\ t \in (0,T]$, find $(u,p) \in (X,M)$, such that for all $(v,q) \in X \times M$

$$\begin{cases} (u_t, v) + a(u,v) - d(v,p) = (f,v), \\ d(u,q) = 0. \end{cases} \tag{2.2}$$

2.1. Finite element approximation

Let $h > 0$ be a real positive parameter. The finite element subspace $X_h \times M_h$ of $X \times M$ is characterized by $\mathcal{T}_h = \mathcal{T}_h(\Omega)$, a partitioning of Ω into triangles K, assumed to be uniformly shape-regular as $h \to 0$, see [10] for details.

We give some examples of the spaces X_h and M_h such that the discrete formula of inf-sup condition (2.1) is satisfied. Let Ω be a convex, polygonal domain in plane and $\mathcal{T}_h$, a partitioning of Ω into triangles K, assumed to be uniformly regular as $h \to 0$. For any nonnegative integer l, we denote by $P_l(K)$ the space of polynomials of degree less than or equal to l on element K.

Example 1. Girault, Raviart [15]

$$X_h = \{v_h \in C^0(\Omega)^2 \cap X;\ v_h \in P_2(K)^2,\ \forall\ K \in \mathcal{T}_h\},$$
$$M_h = \{q_h \in M;\ q_h \in P_0(K),\ \forall K \in \mathcal{T}_h\}.$$

Example 2. Bercovier, Pironneau [7]. We consider the triangulation $\mathcal{T}_{h/2}$ obtained by dividing each triangle of $\mathcal{T}_h$ in four triangles (by joining the mid-sides). We set

$$X_h = \{v_h \in C^0(\overline{\Omega})^2 \cap X;\ v_h \in P_1(K)^2,\ \forall\ K \in \mathcal{T}_{h/2}\},$$
$$M_h = \{q_h \in C^0(\overline{\Omega}) \cap M;\ q_h \in P_1(K),\ \forall K \in \mathcal{T}_h\}.$$

Example 3. Mini-element. We introduce $\widehat{b} \in H_0^1(K)$ taking the value 1 at the barycenter of K and such that $0 \le \widehat{b} \le 1$, which is called a "bubble function". Then, we define the spaces by

$$P_{1,h}^b = \{\phi_h \in C^0(\Omega), \phi_h|_K \in P_1(K) \oplus \mathrm{span}\{\widehat{b}\},\ \forall K \in \mathcal{T}_h\},$$

and

$$X_h = (P_{1,h}^b)^2 \cap X, \quad M_h = \{q_h \in M;\ q_h \in P_1(K),\ \forall K \in \mathcal{T}_h\}.$$

In order to obtain the convergence analysis of the optimal order, we introduce the following definition about Stokes projection.

Definition 2.1. (Stokes projection) For $\forall\ (v,q)\in X\times M$, the projection operators $(R_h,Q_h)\in X_h\times M_h$ is defined by

$$a(R_h,v_h)-d(v_h,Q_h)+d(R_h,q_h)=a(v,v_h)-d(v_h,q)+d(v,q_h) \tag{2.3}$$

for all $(v_h,q_h)\in X_h\times M_h$. Furthermore, for $\forall\ (v,q)\in D(A)\times(H^1(\Omega)\cap M)$, operators (R_h,Q_h) satisfy (see [15, 17]):

$$\|R_h-v\|_0+h(\|\nabla(R_h-v)\|_0+\|Q_h-q\|_0)\leq Ch^2(\|Av\|_0+\|q\|_1). \tag{2.4}$$

Now, we present the space semidiscrete finite element scheme for problem (1.1): Find $(u_h,p_h)\in X_h\times M_h$, such that for all $t\in(0,T]$

$$\begin{cases} (u_{ht},v_h)+a(u_h,v_h)-d(p_h,v_h)=(f,v_h) & \forall\ v_h\in X_h,\\ d(u_h,q_h)=0 & \forall\ q_h\in M_h. \end{cases} \tag{2.5}$$

Remark 2.1. From the continuous and coercivity of $a(\cdot,\cdot)$ on $X_h\times X_h$ with $b(\cdot,\cdot)$ satisfying the discrete inf-sup condition on $X_h\times M_h$, it follows that problem (2.5) admits a unique solution $(u_h,p_h)\in X_h\times M_h$ (see [15, 26]).

We end this section by introducing two important lemmas, which are frequently used in the following analysis (see [9, 10]).

Lemma 2.1. Assume that there exist two L^2-projection operators $I_h: X\to X_h$ and $I_h^n: X\to X_h^n$ (X_h^n will be defined in Section 4) defined by

$$(\phi-I_h^{\square}\phi,w_h)=0\ \ \forall\ w_h\in X_h\ or\ X_h^n,\ \phi\in X,\ \ (I_h^{\square}\ takes\ I_h\ or\ I_h^n).$$

Furthermore, if $\phi\in D(A)$, the following properties hold

$$h^i\|\phi-I_h^{\square}\phi\|_{0,K}\leq Ch_K^{2-i}\|\phi\|_{2,\omega_K}\ (i=0,1),\quad \|\phi-I_h^{\square}\phi\|_{0,E}\leq Ch_E^{1/2}\|\phi\|_{1,\omega_K},$$

where $\omega_K=\cup_{K'\cap K\neq\emptyset}K'$ and $\omega_E=\cup_{E\cap K\neq\emptyset}K$ for $\forall\ K,K'\in\mathcal{T}_h$.

Lemma 2.2. Let $g(t),\ h(t),\ y(t)$ be three locally integrable nonnegative functions on time interval $[0, \infty)$ such that for any fixed time $t_0 \geq 0$ and all $t \geq t_0$

$$y(t) + G(t) \leq C + \int_{t_0}^{t} h(s)ds + \int_{t_0}^{t} g(s)y(s)ds,$$

where $G(t)$ is a nonnegative function on $[0, \infty)$, $C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq (C + \int_{t_0}^{t} h(s)ds) \exp(\int_{t_0}^{t} g(s)ds).$$

3. A posteriori error estimates for semidiscrete formulation

In this section, we present a posteriori error estimates of the space semi-discrete formulation for problem (1.1) and develop the computable upper and lower bounds for numerical solution (u_h, p_h) in various of norms.

Denote $e_u = u_h - u$ and $e_p = p_h - p$. From (2.2) and (2.5), it follows that e_u and e_p satisfy the following error equations

$$\begin{cases} (e_{ut}, v) + a(e_u, v) - d(e_p, v) = r_1(v), \\ d(e_u, q) = r_2(q), \end{cases} \tag{3.1}$$

where the residuals r_1 and r_2 are respectively defined by

$$r_1(v) = (u_{ht}, v) + (\nabla u_h, \nabla v) - (\nabla \cdot v, p_h) - (f, v) \quad \text{and} \quad r_2(q) = (\nabla \cdot u_h, q). \tag{3.2}$$

Furthermore, from equations (2.5), one finds

$$r_1(v_h) = 0, \ \forall \ v_h \in X_h \ \text{ and } \ r_2(q_h) = 0 \ \forall \ q_h \in M_h.$$

We split the errors e_u and e_p into two parts

$$e_u = u_h - u = (\widetilde{u} - u) - (\widetilde{u} - u_h) \triangleq \xi_u - \eta_u,$$

$$e_p = p_h - p = (\widetilde{p} - p) - (\widetilde{p} - p_h) \triangleq \xi_p - \eta_p.$$

Then, (3.1) can be rewritten as

$$\begin{cases} (\xi_{ut}, v) + a(\xi_u, v) - d(\xi_p, v) = r_1(v) + (\eta_{ut}, v) + a(\eta_u, v) - d(\eta_p, v), \\ (\nabla \cdot \xi_u, q) = r_2(q) + (\nabla \cdot \eta_u, q). \end{cases} \tag{3.3}$$

Now, we present the Stokes reconstruction $\widetilde{u} \in X$ and $\widetilde{p} \in M$ of $u_h(t)$ and $p_h(t)$ for all $t \in (0, T]$.

Definition 3.1. (Stokes reconstruction) For given u_h and p_h, define the Stokes reconstruction $\widetilde{u}(t)$ and $\widetilde{p}(t)$ by

$$\begin{cases} (\nabla(\widetilde{u} - u_h), \nabla v) - (\widetilde{p} - p_h, \nabla \cdot v) = -r_1(v), \\ (\nabla \cdot (\widetilde{u} - u_h), q) = -r_2(q). \end{cases} \tag{3.4}$$

For given u_h, p_h, r_1, r_2, it is easy to show that system (3.4) admits a unique solution $(\widetilde{u}, \widetilde{p}) \in X \times M$ for $\forall\ t \in (0, T]$.

By (3.4), equations (3.3) can be rewritten as

$$\begin{cases} (\xi_{ut}, v) + a(\xi_u, v) - d(\xi_p, v) = (\eta_{ut}, v), \\ (\nabla \cdot \xi_u, q) = 0. \end{cases} \tag{3.5}$$

Lemma 3.1. Assume that $\widetilde{u}$ is the solution of Stokes reconstruction (3.4). Then, for $\forall\ q \in M$, $\widetilde{u}$ satisfies

$$\nabla \cdot \widetilde{u} = 0.$$

Proof. From the second equation of (3.5), it follows that

$$0 = (\nabla \cdot \xi_u, q) = (\nabla \cdot (\widetilde{u} - u), q).$$

By the incompressible condition $\mathrm{div} u = 0$, we obtain the desired result.

Lemma 3.2. Assume that Ω is a bounded polygonal domain with a sufficiently smooth boundary $\partial\Omega$. Suppose that ξ_u and ξ_p are solutions of (3.5). Then, for $\forall\ (v, q) \in X \times M$, there exists a constant C such that

$$\|\xi_u(t)\|_0^2 + \int_0^t \|\nabla \xi_u(s)\|_0^2 ds \leq 3\Big(\|\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big), \tag{3.6}$$

$$\Big(\|\nabla \xi_u(t)\|_0^2 + \int_0^t \|\xi_{ut}(s)\|_0^2 ds\Big)^{1/2} \leq \|\nabla \xi_u(0)\|_0 + (\int_0^t \|\eta_{ut}(s)\|_0^2 ds)^{1/2}. \tag{3.7}$$

Moreover,

$$\|\xi_p(t)\|_0 \leq C\Big(\|\nabla \xi_u(0)\|_0 + \|\xi_{ut}(0)\|_0 + \|\eta_{ut}\|_0 + (\int_0^t \|\eta_{utt}(s)\|_0^2 ds)^{\frac{1}{2}} + (\int_0^t \|\eta_{ut}(s)\|_0^2 ds)^{\frac{1}{2}}\Big) \tag{3.8}$$

Proof. Firstly, we choose $v = \xi_u$ and $q = \xi_p$ in (3.5), and add the resulting equations to obtain that

$$\frac{1}{2}\frac{d}{dt}\|\xi_u\|_0^2 + \|\nabla\xi_u\|_0^2 = (\eta_{ut}, \xi_u).$$

Integrating above equation with respect to time from 0 to t and using Cauchy inequality, we obtain

$$\begin{aligned}\|\xi_u(t)\|_0^2 + \int_0^t \|\nabla\xi_u(s)\|_0^2 ds &\leq \|\xi_u(0)\|_0^2 + 2\int_0^t \|\eta_{ut}(s)\|_0 \|\xi_u(s)\|_0 ds \\ &\leq \|\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds + \int_0^t \|\xi_u(s)\|_0^2 ds.\end{aligned}$$

We finish the proof of (3.6) by using Lemma 2.2.

By differentiating (3.5) with respect to time, choosing $v = \xi_{ut},\ q = \xi_{pt}$ and following the proof of (3.6), we have

$$\|\xi_{ut}(t)\|_0^2 + \int_0^t \|\nabla\xi_{ut}(s)\|_0^2 ds \leq 3\Big(\|\xi_{ut}(0)\|_0^2 + \int_0^t \|\eta_{utt}(s)\|_0^2 ds\Big). \tag{3.9}$$

Secondly, differentiating the second equation in (3.5) with respect to time, taking $v = \xi_{ut},\ q = \xi_p$, we obtain by adding the resulting equations that

$$\|\xi_{ut}\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\xi_u\|_0^2 = (\eta_{ut}, \xi_{ut}).$$

Integrating above equation with respect to time from 0 to t, and applying Cauchy inequality, one finds

$$\|\nabla\xi_u(t)\|_0^2 + \int_0^t \|\xi_{ut}(s)\|_0^2 ds \leq \|\nabla\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}\|_0^2 ds.$$

Together with inequality $a^2 + b^2 \leq (a+b)^2$ $(a, b \geq 0)$, we complete the proof (3.7).

Finally, by (2.1) and the first equation of (3.5), we arrive at

$$\begin{aligned}\|\xi_p\|_0 &\le \beta^{-1}\sup_{0\neq v\in X}\frac{|d(\xi_p,q)|}{\|\nabla v\|_0}\\ &\le \beta^{-1}\sup_{0\neq v\in X}\frac{|(\xi_{ut},v)|+|(\nabla\xi_u,\nabla v)|+|(\eta_{ut},v)|}{\|\nabla v\|_0}\\ &\le C(\|\xi_{ut}\|_0+\|\nabla\xi_u\|_0+\|\eta_{ut}\|_0). \qquad (3.10)\end{aligned}$$

The desired result (3.8) follows from (3.7) and (3.9) with (3.10).

3.1. Duality estimates

In this subsection, we like to derive a posteriori error estimates for the transient Stokes equations in spatial semidiscrete scheme (2.5).

Consider the following dual problem: For a fixed time $t^* \in (0,T]$, find $(\Phi,\Psi)\in X\times M$ such that

$$\begin{cases}\Phi_t+\Delta\Phi-\nabla\Psi=0 & \text{in } \Omega\times(0,t^*],\\ \nabla\cdot\Phi=0 & \text{in } \Omega\times(0,t^*],\\ \Phi=0 & \text{on } \partial\Omega\times(0,t^*],\\ \Phi(t^*)=\xi_u(t^*) & \text{in } \Omega,\end{cases} \qquad (3.11)$$

where $\xi_u=\widetilde{u}-u$. Multiplying the first equation of (3.11) by Φ and the second equation by Ψ, integrating with respect to time from 0 to t^*, one finds

$$\|\xi_u(t^*)\|_0^2=\|\Phi(0)\|_0^2+\int_0^{t^*}\|\nabla\Phi(s)\|_0^2ds. \qquad (3.12)$$

Meanwhile, multiplying the first equation of (3.11) by ξ_u and the second equation by ξ_p, then integrating over $\Omega\times(0,t^*]$ and using Green's formula, we deduce that

$$\|\xi_u(t^*)\|_0^2=(\Phi(0),\xi_u(0))+\int_0^{t^*}\Big((\Phi,\xi_{ut})+(\nabla\Phi,\nabla\xi_u)-(\nabla\cdot\xi_u,\Psi)\Big)dt \qquad (3.13)$$

and

$$\int_0^{t^*}(\nabla\cdot\Phi,\xi_p)dt=0. \qquad (3.14)$$

Adding (3.13) and (3.14), regrouping these terms and using (3.5) with $v = \Phi, q = \Psi$ and the estimate $(\eta_{ut}, \Phi) \leq \|\eta_{ut}\|_0 \|\Phi\|_0$, one derives

$$\begin{aligned} \|\xi_u(t^*)\|_0^2 &= (\Phi(0), \xi_u(0)) + \int_0^{t^*} (\eta_{ut}, \Phi) dt \\ &\leq \|\Phi(0)\|_0 \|\xi_u(0)\|_0 + (\int_0^{t^*} \|\eta_{ut}\|_0^2 ds)^{\frac{1}{2}} (\int_0^{t^*} \|\Phi(s)\|_0^2 ds)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

By (3.12) and the fact that $\|\Phi\|_0 \leq \|\nabla\Phi\|_0$ (for $\forall\ \Phi \in X$) in (3.15), we obtain that

$$\|\xi_u(t^*)\|_0 \leq \|\xi_u(0)\|_0 + (\int_0^{t^*} \|\eta_{ut}\|_0^2 ds)^{\frac{1}{2}}. \tag{3.16}$$

which verifies our result (3.6).

For the $L^2(L^2)$ estimates about ξ_p, we consider the following adjoint problem

$$\begin{cases} \Phi_t + \Delta\Phi - \nabla\Psi = 0 & \text{in } \Omega \times (0, t^*], \\ -\nabla \cdot \Phi = \xi_p & \text{in } \Omega \times (0, t^*], \\ \Phi = 0 & \text{on } \partial\Omega \times (0, t^*], \\ \Phi(t^*) = 0 & \text{in } \Omega, \end{cases} \tag{3.17}$$

where $\xi_p = \widetilde{p} - p$. Assume that $\Phi \in L^2(0, T; H_0^1(\Omega)^2)$ and $\Phi_t \in L^2(0, T; L^2(\Omega)^2)$. We can obtain that Φ is bounded, i.e., there exists a constant C such that

$$\max_{t \in (0,T]} \|\Phi(t)\|_0 \leq C. \tag{3.18}$$

Multiplying the first equation of (3.17) by ξ_u and the second equation by ξ_p respectively, we obtain

$$\frac{d}{dt}(\Phi, \xi_u) - (\Phi, \xi_{ut}) - (\nabla\Phi, \nabla\xi_u) + (\Psi, \nabla \cdot \xi_u) = 0 \tag{3.19}$$

and

$$\|\xi_p\|_0^2 = -(\nabla \cdot \Phi, \xi_p). \tag{3.20}$$

Adding equations (3.19) and (3.20), regrouping these terms and using (3.5) with $v = \Phi, q = \Psi$, we derive that

$$\|\xi_p\|_0^2 = (\eta_{ut}, \Phi) - \frac{d}{dt}(\Phi, \xi_u). \tag{3.21}$$

Integrating (3.21) with respect to time from 0 to t^* ($t^* \leq T$), and using stability result (3.18), one finds

$$\int_0^{t^*} \|\xi_p(s)\|_0^2 ds \leq C\Big(\|\xi_u(0)\|_0 + \int_0^{t^*} \|\eta_{ut}(s)\|_0 ds\Big).$$

3.2. A posteriori error estimates for the Stokes reconstruction

In this subsection, we discuss a posteriori error estimates for the unsteady Stokes problem. To achieve this aim, we need some a posteriori estimates about $\eta_u, \eta_{ut}, \nabla\eta_u$ and η_p related to the Stokes reconstruction (3.4).

Lemma 3.3. Assume that Ω is a bounded polygonal domain with sufficiently smooth boundary. Suppose that $\widetilde{u}$ and $\widetilde{p}$ are solutions of (3.4), under the assumption of (**A1**). Then, there exists a constant C, independent of mesh parameter h, such that

$$\begin{aligned}
\|\eta_u\|_0 \leq\ & C\Big(\big(\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nabla u_h - p_h \cdot \mathbf{I}]\|_0^2\big)^{1/2} + \big(\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla\cdot u_h\|_{0,K}^2\big)^{1/2} \\
& + \big(\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2\big)^{1/2}\Big), \qquad (3.22) \\
\|\eta_p(t)\|_0 + \|\nabla\eta_u\|_0 \leq\ & C\Big(\big(\sum_{E\in\mathcal{E}_h} h_E \|[\nabla u_h - p_h \cdot \mathbf{I}]\|_0^2\big)^{1/2} + \big(\sum_{K\in\mathcal{T}_h} \|\nabla\cdot u_h\|_{0,K}^2\big)^{1/2} \\
& + \big(\sum_{K\in\mathcal{T}_h} h_K^2 \|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2\big)^{1/2}\Big). \qquad (3.23)
\end{aligned}$$

where $\mathbf{I}$ is a 2-dimension unit matrix.

Proof. Firstly, we discuss the estimate of η_u in L^2-norm based on the Aubin-Nitsche duality arguments. Consider $\Phi \in D(A)$, $\Psi \in H^1(\Omega) \cap M$ as the solutions of the elliptic problem

$$\begin{cases} -\Delta\Phi + \nabla\Psi = g & \text{in } \Omega, \\ \nabla\cdot\Phi = 0 & \text{in } \Omega. \end{cases} \tag{3.24}$$

Assume that the following elliptic regularity result holds

$$\|\Phi\|_2 + \|\Psi\|_1 \le C\|g\|_0. \tag{3.25}$$

Multiplying the first equation of (3.24) by $\eta_u = \widetilde{u} - u_h$ and the second equation by $\eta_p = \widetilde{p} - p_h$ respectively, adding and regrouping them, using (2.3), we have

$$\begin{aligned} & (\eta_u, g) \\ = \;& (\nabla\Phi, \nabla\widetilde{u}) - (\Psi, \nabla\cdot\widetilde{u}) - (\nabla\cdot\Phi, \widetilde{p}) - (\nabla\Phi, \nabla u_h) + (\Psi, \nabla\cdot u_h) + (\nabla\cdot\Phi, p_h) \\ = \;& (\nabla\Phi, \nabla\widetilde{u}) - (\Psi, \nabla\cdot\widetilde{u}) - (\nabla\cdot\Phi, \widetilde{p}) - (\nabla R_h, \nabla u_h) + (Q_h, \nabla\cdot u_h) + (\nabla\cdot R_h, p_h) \\ = \;& (\nabla(\Phi - R_h), \nabla\widetilde{u}) - (\Psi - Q_h, \nabla\cdot\widetilde{u}) - (\nabla\cdot(\Phi - R_h), \widetilde{p}) \\ & + (\nabla R_h, \nabla(\widetilde{u} - u_h)) - (Q_h, \nabla\cdot(\widetilde{u} - u_h)) - (\nabla\cdot R_h, \widetilde{p} - p_h). \end{aligned} \tag{3.26}$$

From the definition of (3.4), we know that

$$(\nabla R_h, \nabla(\widetilde{u} - u_h)) - (\nabla\cdot R_h, \widetilde{p} - p_h) = -r_1(R_h),$$

and

$$-(Q_h, \nabla\cdot(\widetilde{u} - u_h)) = r_2(Q_h).$$

According to the expressions in (3.2) about $r_1(v)$ and $r_2(q)$ with $v = R_h, q = Q_h$, and using (2.5), equation (3.26) can be rewritten as

$$(\eta_u, g) = (\nabla(\Phi - R_h), \nabla\widetilde{u}) - (\Psi - Q_h, \nabla\cdot\widetilde{u}) - (\nabla\cdot(\Phi - R_h), \widetilde{p}).$$

Applying (2.3) again and noting (3.4), we deduce that

$$\begin{aligned} & (\eta_u, g) \\ = \;& (\nabla(\Phi - R_h), \nabla\widetilde{u}) - (\Psi - Q_h, \nabla\cdot\widetilde{u}) - (\nabla\cdot(\Phi - R_h), \widetilde{p}) \\ = \;& (\nabla(\Phi - R_h), \nabla(\widetilde{u} - u_h)) - (\Psi - Q_h, \nabla\cdot(\widetilde{u} - u_h)) - (\nabla\cdot(\Phi - R_h), (\widetilde{p} - p_h)) \\ = \;& r_2(\Psi - Q_h) - r_1(\Phi - R_h). \end{aligned} \tag{3.27}$$

Integrating by parts and using (2.4), one finds

$$\begin{aligned}
(\eta_u, g) &= (u_{ht} - \Delta u_h + \nabla p_h - f, \Phi - R_h) \\
&\quad + \sum_{E\in\mathcal{E}_h} \int_E [\nabla u_h - p_h \cdot \mathbf{I}] \cdot (\Phi - R_h) ds + (\nabla \cdot u_h, \Psi - Q_h) \\
&\le \sum_{K\in\mathcal{T}_h} \|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K} \|\Phi - R_h\|_{0,K} \\
&\quad + \sum_{E\in\mathcal{E}_h} \|[\nabla u_h - p_h \cdot \mathbf{I}]\|_{0,E} \|\Phi - R_h\|_{0,E} + \sum_{K\in\mathcal{T}_h} \|\nabla \cdot u_h\|_{0,K} \|\Psi - Q_h\|_{0,K} \\
&\le C\Big((\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2)^{\frac{1}{2}} + (\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nabla u_h - p_h \cdot \mathbf{I}]\|_{0,E}^2)^{\frac{1}{2}} \\
&\quad + (\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla \cdot u_h\|_{0,K}^2)^{\frac{1}{2}} \Big) \cdot (\|\Phi\|_2 + \|\Psi\|_1). \qquad (3.28)
\end{aligned}$$

By elliptic regularity (3.25) in (3.28), we deduce that

$$\begin{aligned}
\frac{(\eta_u, g)}{\|g\|_0} &\le C\Big((\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2)^{\frac{1}{2}} \\
&\quad + (\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nabla u_h - p_h \cdot \mathbf{I}]\|_{0,E}^2)^{\frac{1}{2}} + (\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla \cdot u_h\|_{0,K}^2)^{\frac{1}{2}} \Big). \qquad (3.29)
\end{aligned}$$

Taking the supremum over g, we obtain the desired result (3.22).

By differentiating (3.27) with respect to time one or two times and following the proofs of (3.22), we can obtain the following estimates.

$$\begin{aligned}
\|\eta_{ut}\|_0 &\le C\Big((\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nabla u_{ht} - p_{ht} \cdot \mathbf{I}]\|_0^2)^{1/2} + (\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla \cdot u_{ht}\|_{0,K}^2)^{1/2} \\
&\quad + (\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{htt} - \Delta u_{ht} + \nabla p_{ht} - f_t\|_{0,K}^2)^{1/2} \Big), \qquad (3.30) \\
\|\eta_{utt}\|_0 &\le C\Big((\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nabla u_{htt} - p_{htt} \cdot \mathbf{I}]\|_0^2)^{1/2} + (\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla \cdot u_{htt}\|_{0,K}^2)^{1/2} \\
&\quad + (\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{httt} - \Delta u_{htt} + \nabla p_{htt} - f_{tt}\|_{0,K}^2)^{1/2} \Big). \qquad (3.31)
\end{aligned}$$

For the estimate of (3.23), choosing $v = \eta_u,\ q = \eta_p$ in (3.4), and using the fact that $r_1(I_h\eta_u) = 0$, thanks to Green's formula and Lemma 3.2, we obtain that

$$\begin{aligned}
\|\nabla\eta_u\|_0^2 &= (\nabla\cdot\eta_u,\eta_p) - r_1(\eta_u) = -r_1(\eta_u) - r_2(\eta_p) = -r_1(\eta_u - I_h\eta_u) - r_2(\eta_p) \\
&= (u_{ht} - \Delta u_h + \nabla p_h - f, \eta_u - I_h\eta_u) \\
&\quad + \sum_{E\in\mathcal{E}_h}\int_E [\nabla u_h - p_h\cdot\mathbf{I}]\cdot(\eta_u - I_h\eta_u)ds + (\nabla\cdot u_h,\eta_p) \\
&\le C\Big((\sum_{K\in\mathcal{T}_h} h_K^2\|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2)^{\frac{1}{2}}\|\nabla\eta_u\|_0 \\
&\quad +(\sum_{E\in\mathcal{E}_h} h_E\|[\nabla u_h - p_h\cdot\mathbf{I}]\|_{0,E}^2)^{\frac{1}{2}}\|\nabla\eta_u\|_0 + (\sum_{K\in\mathcal{T}_h}\|\nabla\cdot u_h\|_{0,K}^2)^{\frac{1}{2}}\|\eta_p\|_0\Big).
\end{aligned}$$

Combining with Cauchy inequality, one finds

$$\begin{aligned}
\|\nabla\eta_u\|_0^2 &\le C_1\Big((\sum_{K\in\mathcal{T}_h} h_K^2\|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2) + (\sum_{E\in\mathcal{E}_h} h_E\|[\nabla u_h - p_h\cdot\mathbf{I}]\|_{0,E}^2)\Big) \\
&\quad +C_2(\sum_{K\in\mathcal{T}_h}\|\nabla\cdot u_h\|_{0,K}^2)^{\frac{1}{2}}\|\eta_p\|_0. \qquad (3.32)
\end{aligned}$$

Finally, thanks to the inf-sup condition (2.1) and the Stokes reconstruction (3.4), using the fact $r_1(I_hv) = 0$ again, we conclude that

$$\begin{aligned}
\|\eta_p\|_0 &\le \beta^{-1}\sup_{0\neq v\in X}\frac{|(\nabla\cdot v,\eta_p)|}{\|\nabla v\|_0}\beta^{-1}\sup_{0\neq v\in X} \le \frac{|(\nabla\eta_u,\nabla v)| + |r_1(v)|}{\|\nabla v\|_0} \\
&\le \beta^{-1}\sup_{0\neq v\in X}\frac{|(\nabla\eta_u,\nabla v)| + |r_1(v - I_hv)|}{\|\nabla v\|_0} \\
&\le C\Big(\|\nabla\eta_u\|_0 + (\sum_{K\in\mathcal{T}_h} h_K^2\|u_{ht} - \Delta u_h + \nabla p_h - f\|_{0,K}^2)^{\frac{1}{2}} \\
&\quad +(\sum_{E\in\mathcal{E}_h} h_E\|[\nabla u_h - p_h\cdot\mathbf{I}]\|_{0,E}^2)^{\frac{1}{2}}\Big). \qquad (3.33)
\end{aligned}$$

Combining (3.32) with (3.33), we finish the proof of (3.23).

Using Lemmas 3.2 and 3.3, we finally obtain the main theorem of this section.

Theorem 3.4. Let Ω be a bounded polygonal domain with a sufficiently smooth boundary. Assume that (u,p) and (u_h,p_h) are the solutions of (2.2) and (2.5), respectively. Then, under the assumption of (**A1**), for all $t \in (0,T]$, there exists a constant C such that

$$\|u_h - u\|_0^2 \le C\Big(\|u_h(0) - u_0\|_0^2 + \|\eta_u(0)\|_0^2 + \|\eta_u\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big),$$

$$\|\nabla(u_h - u)\|_0^2 \le C\Big(\|\nabla(u_h(0) - u_0)\|_0^2 + \|\nabla\eta_u(0)\|_0^2 + \|\nabla\eta_u\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big),$$

$$\begin{aligned}\|p_h - p\|_0 \le C\Big(&\|\nabla(u_h(0) - u_0)\|_0 + \|\nabla\eta_u(0)\|_0 + \|u_{ht}(0) - u_t(0)\|_0 + \|\eta_{ut}(0)\|_0 \\ &+\|\eta_{ut}\|_0 + \|\eta_p\|_0 + (\int_0^t \|\eta_{ut}(s)\|_0^2 ds)^{1/2} + (\int_0^t \|\eta_{utt}(s)\|_0^2 ds)^{1/2}\Big),\end{aligned}$$

where the estimates of $\eta_u, \eta_{ut}, \eta_{utt}$ and η_p are given by (3.22), (3.30), (3.31) and (3.23), respectively.

4. A posteriori estimates for fully discrete scheme

In this section, we consider a posteriori error estimates of fully discrete approximation for transient Stokes equations (1.1) based on backward Euler method.

Set $0 = t_0 < t_1 < \cdots < t_N = T,\ I_n = (t_{n-1}, t_n]$ and denote $k_n = t_n - t_{n-1}$. For $\forall\, n \in [0,N]$, let $\mathcal{T}_n$ be a refinement of macrotriangulation which is a triangulation of the domain Ω that satisfies the same conformity and shape regularity assumptions made on its refinements (see [9] for details). Denote

$$h_n(x) = \mathrm{diam}(K), \quad \text{where } K \in \mathcal{T}_n \text{ and } x \in K.$$

Given two compatible triangulations $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$, namely, they are refinements of the same macrotriangulation, set $\hat{\mathcal{T}}_n$ be the finest common coarsening of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$, whose mesh size is given by $\hat{h}_n = \max(h_n, h_{n-1})$, for more information about the triangulations, please see Appendix A of [20].

Denote

$$\partial_t \phi^n := \frac{1}{k_n}(\phi^n - \phi^{n-1}), \qquad f^n = f(t_n).$$

We consider X_h^n and M_h^n defined over the triangulations $\mathcal{T}_n$ as the finite element subspaces of X and M, respectively.

Given $U^0 = I_h^0 u_0$, find $\{(U^n, P^n)\}$ with $(U^n, P^n) \in X_h^n \times M_h^n$, for $\forall\ (v, q) \in X_h^n \times M_h^n$. For $n = 0$

$$\begin{cases} (\nabla U^0, \nabla v) - (\nabla \cdot v, P^0) = (f^0, v), \\ (\nabla \cdot U^0, q) = 0, \end{cases} \tag{4.1}$$

and for $n \in [1 : N]$

$$\begin{cases} \frac{1}{k_n}(U^n - U^{n-1}, v) + (\nabla U^n, \nabla v) - (\nabla \cdot v, P^n) = (f^n, v), \\ (\nabla \cdot U^n, q) = 0. \end{cases} \tag{4.2}$$

Using a sequence of discrete values $\{U^n\}, n = 0, 1, 2, \dots, N$, for $\forall\ t \in (0, T]$ we define a continuous piecewise linear function $U(t)$ by

$$U(t) = (1 - \frac{t - t_{n-1}}{k_n})U^{n-1} + \frac{t - t_{n-1}}{k_n}U^n, \quad t_{n-1} < t \le t_n,\ n = 1, 2, \dots, N. \tag{4.3}$$

Similarly, we define $P(t)$ from the set of values $\{P^n\},\ n = 0, 1, 2, \dots, N$, as

$$P(t) = (1 - \frac{t - t_{n-1}}{k_n})P^{n-1} + \frac{t - t_{n-1}}{k_n}P^n, \quad t_{n-1} < t \le t_n,\ n = 1, 2, \dots, N. \tag{4.4}$$

Note that the time derivative of U restricted to I_n is

$$U_t|_{I_n} = \partial_t U^n \qquad \text{for } \forall\ t \in I_n. \tag{4.5}$$

To motivate the use of the Stokes reconstruction, we denote $e_u(t) = U(t) - u(t)$ and $e_p(t) = P(t) - p(t)$. Then, for $\forall\ (v, q) \in X_h^n \times M_h^n$, the pair (e_u, e_p) satisfies

$$\begin{cases} (e_{ut}, v) + (\nabla e_u, \nabla v) - (\nabla \cdot v, e_p) = (\nabla(U - U^n), \nabla v) - (\nabla \cdot v, P - P^n) \\ \qquad\qquad + (f^n - f, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) + \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v) \\ \qquad\qquad + (\nabla U^n, \nabla v) - (\nabla \cdot v, P^n) - (f^n, v), \\ (\nabla \cdot e_u, q) = (\nabla \cdot (U - U^n), q) + (\nabla \cdot U^n, q). \end{cases} \tag{4.6}$$

Define the residuals r_1^n and r_2^n for $n = 1, 2, \dots, N$ by

$$r_1^n(v) = \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v) + (\nabla U^n, \nabla v) - (\nabla \cdot v, P^n) - (f^n, v), \tag{4.7}$$

and

$$r_2^n(q) = (\nabla \cdot U^n, q). \tag{4.8}$$

Now, we present the Stokes reconstruction at time level $t = t_n$ in fully discrete scheme for problem (1.1).

Definition 4.1. (Stokes reconstruction) For given U^n and P^n, $n = 0, 1, \ldots, N$, and $\forall\ (v, q) \in X \times M$, find $\widetilde{u}^n \in X$ and $\widetilde{p}^n \in M$ satisfying

$$\begin{cases} (\nabla(\widetilde{u}^n - U^n), \nabla v) - (\nabla \cdot v, \widetilde{p}^n - P^n) = -r_1^n(v), \\ (\nabla \cdot (\widetilde{u}^n - U^n), q) = -r_2^n(q). \end{cases} \tag{4.9}$$

According to (4.7) and (4.8), thanks to (4.2) and Lemma 2.1, there is

$$r_1^n(v_h) = 0 \text{ for } \forall\ v_h \in X_h^n, \quad \text{and} \quad r_2(q_h) = 0 \text{ for } \forall\ q_h \in M_h^n.$$

Note that $(\widetilde{u}^n, \widetilde{p}^n)$ are the Stokes reconstruction of (U^n, P^n) at $t = t_n$. Using a sequence of discrete values $\{\widetilde{u}^n\}$ $(n = 0, 1, \ldots, N)$, for $\forall\ t \in [0, T]$ we define a continuous function of time as the continuous piecewise linear interpolation $\widetilde{u}(t)$:

$$\widetilde{u}(t) = (1 - \frac{t - t_{n-1}}{k_n})\widetilde{u}^{n-1} + \frac{t - t_{n-1}}{k_n}\widetilde{u}^n, \quad t_{n-1} < t \leq t_n,\ n = 1, 2, \ldots, N. \tag{4.10}$$

Similarly, we define $\widetilde{p}(t)$ from the set of values $\{\widetilde{p}^n\}$, $(n = 0, 1, 2, \ldots, N)$ as

$$\widetilde{p}(t) = (1 - \frac{t - t_{n-1}}{k_n})\widetilde{p}^{n-1} + \frac{t - t_{n-1}}{k_n}\widetilde{p}^n, \quad t_{n-1} < t \leq t_n,\ n = 1, 2, \ldots, N. \tag{4.11}$$

Furthermore, for any $t \in [0, T]$, $(v, q) \in X \times M$, the functions $\widetilde{u}$ and $\widetilde{p}$ satisfy:

$$\begin{cases} (\nabla(\widetilde{u} - U), \nabla v) - (\nabla \cdot v, \widetilde{p} - P) = -r_1(v), \\ (\nabla \cdot (\widetilde{u} - U), q) = -r_2(q), \end{cases} \tag{4.12}$$

where r_1 and r_2 are piecewise linear interpolations of $\{r_1^n\}_{n=1}^N$ and $\{r_2^n\}_{n=1}^N$, respectively. We split the errors of e_u and e_p into two parts

$$e_u = (\widetilde{u} - u) - (\widetilde{u} - U) \triangleq \xi_u - \eta_u, \tag{4.13}$$

$$e_p = (\widetilde{p} - p) - (\widetilde{p} - P) \triangleq \xi_p - \eta_p. \tag{4.14}$$

Thanks to (4.7)-(4.9), for $\forall\ (v,q) \in X \times M$, equations (4.6) can be rewritten as

$$\begin{cases} (\xi_{ut}, v) + (\nabla \xi_u, \nabla v) - (\nabla \cdot v, \xi_p) = (\eta_{ut}, v) + (\nabla(\widetilde{u} - \widetilde{u}^n), \nabla v) \\ \qquad -(\nabla \cdot v, \widetilde{p} - \widetilde{p}^n) + (f^n - f, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v), \\ (\nabla \cdot \xi_u, q) = (\nabla \cdot (\widetilde{u} - \widetilde{u}^n), q). \end{cases} \tag{4.15}$$

Note that

$$\widetilde{u} - \widetilde{u}^n = -\frac{t_n - t}{k_n}(\widetilde{u}^n - \widetilde{u}^{n-1}) = -(t_n - t)\partial_t \widetilde{u}^n \tag{4.16}$$

and

$$\widetilde{p} - \widetilde{p}^n = -\frac{t_n - t}{k_n}(\widetilde{p}^n - \widetilde{p}^{n-1}) = -(t_n - t)\partial_t \widetilde{p}^n. \tag{4.17}$$

By (4.16), the second equation in (4.15) can be transformed into

$$(\nabla \cdot \xi_u, q) = (\nabla \cdot (\widetilde{u} - \widetilde{u}^n), q) = -\frac{t_n - t}{k_n}\Big(\nabla \cdot (\widetilde{u}^n - \widetilde{u}^{n-1}), q\Big). \tag{4.18}$$

From the definition of the Stokes reconstruction (4.9), it follows that

$$(\nabla \cdot \widetilde{u}^n, q) = (\nabla \cdot U^n, q) - r_2^n(q) \quad \text{ for } n = 0, 1, \ldots, N. \tag{4.19}$$

Combining (4.8) and (4.19), one finds

$$(\nabla \cdot \widetilde{u}^n, q) = 0 \qquad \text{for } n = 0, 1, \ldots, N. \tag{4.20}$$

With the help of (4.20), equation (4.18) can be simplified as

$$(\nabla \cdot \xi_u, q) = 0. \tag{4.21}$$

From the first equation of (4.9), it can be deduced that

$$(\nabla \widetilde{u}^n, \nabla v) - (\nabla \cdot v, \widetilde{p}^n) = (\nabla U^n, \nabla v) - (\nabla \cdot v, P^n) - r_1^n(v). \tag{4.22}$$

As a consequence, we arrive at

$$\begin{aligned} & (\nabla(\widetilde{u}^n - \widetilde{u}^{n-1}), \nabla v) - (\nabla \cdot v, (\widetilde{p}^n - \widetilde{p}^{n-1})) \\ = \ & (\nabla(U^n - U^{n-1}), \nabla v) - (\nabla \cdot v, P^n - P^{n-1}) - (r_1^n(v) - r_1^{n-1}(v)). \end{aligned} \tag{4.23}$$

Substituting (4.23) into the first equation of (4.15) and combining with (4.21), we obtain that

$$\begin{cases} (\xi_{ut}, v) + (\nabla\xi_u, \nabla v) - (\nabla\cdot v, \xi_p) = (\eta_{ut}, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) + (f^n - f, v) \\ \quad -\frac{t_n - t}{k_n}\Big((r_1^n(v) - r_1^{n-1}(v)) - (\nabla(U^n - U^{n-1}), \nabla v) + (\nabla\cdot v, P^n - P^{n-1})\Big), \\ (\nabla\cdot\xi_u, q) = 0. \end{cases} \tag{4.24}$$

Now, we present the estimates of errors between the Stokes reconstruction $(\widetilde{u}, \widetilde{p})$ and the exact solution (u, p) in various norms. In order to simply the expressions, we introduce some notations. Set

$$\mathscr{E}_1^m = \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|f^n - f\|_0^2 ds, \qquad \mathscr{E}_2^m = \sum_{n=1}^m k_n \|k_n^{-1} h_n (I - I_h^n) U^{n-1}\|_0^2,$$

$$\mathscr{E}_3^m = \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\eta_{ut}(s)\|_0^2 ds, \qquad \mathscr{E}_4^m = \sum_{n=1}^m k_n^3 \Big(\|\hat{h}_n \partial_t r_1^n\|_0^2 + \|\nabla\partial_t U^n\|_0^2 + \|\partial_t P^n\|_0^2\Big),$$

$$\mathscr{E}_5^m = \|h_1(I_h^1 - I)(\frac{1}{k_1}U^0)\|_0 + \sum_{n=2}^m k_n \|\hat{h}_n \partial_t (I_h^n - I)(\frac{1}{k_n}U^{n-1})\|_0 + \|h_m(I_h^m - I)(\frac{1}{k_m}U^{m-1})\|_0,$$

$$\mathscr{E}_6^m = \sum_{n=1}^m k_n^3 (\|\partial_t r_1^n\|_0^2 + \|\Delta\partial_t U^n\|_0^2 + \|\nabla\partial_t P^n\|_0^2) \qquad \mathscr{E}_7^m = \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\eta_{utt}(s)\|_0^2 ds.$$

Theorem 4.1. Let Ω be a bounded convex polygonal domain, Assume that (u, p) and $(\widetilde{u}, \widetilde{p})$ are the solutions of (2.2) and (4.9), respectively. Under the assumption of (**A1**), for $m \in [1, N]$, there exists a constant C such that

$$\|u(t_m) - \widetilde{u}(t_m)\|_0^2 + \int_0^{t_m} \|\nabla(u - \widetilde{u})\|_0^2 ds \le \|e_u(0)\|_0^2 + \|\eta_u(0)\|_0^2 + 3\sum_{i=1}^4 \mathscr{E}_i^n, \tag{4.25}$$

$$\Big(\|\nabla(u - \widetilde{u})\|_0^2 + \int_0^{t_j} \|u_t - \widetilde{u}_t\|_0^2 ds\Big)^{\frac{1}{2}} \le \Big(\mathscr{E}_1^n + \mathscr{E}_3^n + \mathscr{E}_6^n\Big)^{1/2} + C\mathscr{E}_5^n, \tag{4.26}$$

$$\begin{aligned} \|p - \widetilde{p}\|_0 \quad \le \quad & \|e_{ut}(0)\|_0 + \|\eta_{ut}(0)\|_0 + \|\eta_{ut}\|_0 + \|h_n k_n^{-1}(I - I_h^n)U^{n-1}\|_0 + C\mathscr{E}_5^n + \|f - f^n\|_0 \\ & + \Big(\mathscr{E}_1^n + \mathscr{E}_3^n + \mathscr{E}_6^n + \mathscr{E}_7^n\Big)^{1/2} + k_n\Big(\|h_n\partial_t r_1^n\|_0 + \|\nabla\partial_t U^n\|_0 + \|\partial_t P^n\|_0\Big). \end{aligned} \tag{4.27}$$

Proof. Choosing $v = \xi_u$ in the first equations of (4.24) and $q = \xi_p$ in the second equation, adding the resulting equations and using Lemma 2.1, we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi_u\|_0^2 + \|\nabla\xi_u\|_0^2 &= (\eta_{ut}, \xi_u) + (f^n - f, \xi_u) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_u) \\
&\quad -(t_n - t)\Big(\partial_t r_1^n(\xi_u) - (\nabla\partial_t U^n, \nabla\xi_u) + (\nabla\cdot\xi_u, \partial_t P^n)\Big) \\
&= (\eta_{ut}, \xi_u) + (f^n - f, \xi_n) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_u - I_h^n\xi_u) \\
&\quad -(t_n - t)\Big(\partial_t r_1^n(\xi_u) - (\nabla\partial_t U^n, \nabla\xi_u) + (\nabla\cdot\xi_u, \partial_t P^n)\Big) \\
&\triangleq T_1^n + T_2^n + T_3^n + T_4^n. \qquad (4.28)
\end{aligned}$$

Now, we estimate the right-hand side terms of (4.28) separately. For T_1^n and T_2^n, with Cauchy inequality, it is easy to see that

$$\begin{aligned}
|(\eta_{ut}, \xi_u)| + |(f^n - f, \xi_n)| &\le \|\eta_{ut}\|_0\|\xi_u\|_0 + \|f^n - f\|_0\|\xi_u\|_0 \\
&\le \frac{3}{2}\Big(\|\eta_{ut}\|_0^2 + \|f^n - f\|_0^2\Big) + \frac{1}{6}\|\nabla\xi_u\|_0^2.
\end{aligned}$$

For T_3^n, by Cauchy inequality and Lemma 2.1, we find that

$$\begin{aligned}
|T_3^n| &\le k_n^{-1}\|I_h^n U^{n-1} - U^{n-1}\|_0\|\xi_u - I_h^n\xi_u\|_0 \\
&\le \frac{3}{2}\|k_n^{-1}h_n(I - I_h^n)U^{n-1}\|_0^2 + \frac{1}{6}\|\nabla\xi_u\|_0^2.
\end{aligned}$$

For T_4^n, using the fact that $r_1^n(v_h) = 0$ for $\forall\ v_h \in X_h$, we have $(r_1^n - r_1^{n-1})(v_h) = 0$ for all $v_h \in X^n \cap X^{n-1}$. Let $\hat{I}_h^n$ be the L^2-projection relative to the finest common coarsening $\hat{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$, for $\forall\ t \in (t_{n-1}, t_n]$. we deduce that

$$\begin{aligned}
|T_4^n| &\le (t_n - t)\Big(\partial_t r_1^n(\xi_u - \hat{I}_h^n\xi_u) - (\nabla\partial_t U^n, \nabla\xi_u) + (\nabla\cdot\xi_u, \partial_t P^n)\Big) \\
&\le \frac{3}{2}k_n^2\Big(\|\hat{h}_n\partial_t r_1^n\|_0^2 + \|\nabla\partial_t U^n\|_0^2 + \|\partial_t P^n\|_0^2\Big) + \frac{1}{6}\|\nabla\xi_u\|_0^2,
\end{aligned}$$

Combining above inequalities with (4.28), integrating with respect with time from 0 to t_m with $m \in [1 : N]$, we complete the proof of (4.25).

Differentiating (4.24) with respect to time with $v = \xi_{ut},\ q = \xi_{pt}$. Following the proofs of (4.25) and using (4.7) and Taylor expansion, we deduce that

$$
\begin{aligned}
&\|\xi_{ut}(t_m)\|_0^2 + \int_0^{t_m} \|\nabla \xi_{ut}(s)\|_0^2 ds \\
\leq\ & C\Big[\|e_{ut}(0)\|_0^2 + \|\eta_{ut}(0)\|_0^2 + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} \|\eta_{utt}(s)\|_0^2 ds \\
&+ \sum_{n=1}^{m} k_n\Big(\|\hat{h}_n U_t^{n-1}\|_0^2 + \|f^n - f^{n-1}\|_0^2 + \|f_t^n - f_t\|_0^2\Big)\Big]. \qquad (4.29)
\end{aligned}
$$

Next, we provide the proof of (4.26). Choosing $v = \xi_{ut}$ in the first equation of (4.24) and differentiating the second equation with respect to time and taking $q = \xi_p$, we add the resulting equations to yield

$$
\begin{aligned}
\|\xi_{ut}\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \xi_u\|_0^2 &= (\eta_{ut}, \xi_{ut}) + (f^n - f, \xi_{ut}) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_{ut}) \\
&\quad -(t_n - t)\Big(\partial_t r_1^n(\xi_{ut}) - (\nabla \partial_t U^n, \nabla \xi_{ut}) + (\nabla \cdot \xi_{ut}, \partial_t P^n)\Big) \\
&= (\eta_{ut}, \xi_{ut}) + (f^n - f, \xi_{ut}) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_{ut} - I_h^n \xi_{ut}) \\
&\quad -(t_n - t)\Big(\partial_t r_1^n(\xi_{ut}) - (\nabla \partial_t U^n, \nabla \xi_{ut}) + (\nabla \cdot \xi_{ut}, \partial_t P^n)\Big) \\
&\triangleq T_1^n + T_2^n + T_3^n + T_4^n. \qquad (4.30)
\end{aligned}
$$

Integrating (4.30) with respect to time from 0 to t_m for any $m \in [1 : N]$, one gets

$$
\begin{aligned}
&\|\nabla \xi_u(t_m)\|_0^2 + 2\int_0^{t_m} \|\xi_{ut}\|_0^2 ds \\
=\ & \|\nabla \xi_u(0)\|_0^2 + 2\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} (T_1^n + T_2^n + T_3^n + T_4^n) ds. \qquad (4.31)
\end{aligned}
$$

Set

$$
\mathscr{F}^2(t_j) = \max_{0 \leq t_j \leq t_m} \Big(\|\nabla \xi_u(t_m)\|_0^2 + \int_0^{t_j} \|\xi_{ut}\|_0^2 ds\Big), \quad \text{where } j \in [0 : m]. \qquad (4.32)
$$

In terms of the Cauchy inequality and Green's formula, we obtain that

$$
\begin{aligned}
& T_1^n + T_2^n + T_4^n \\
\leq \; & \Big[\|\eta_{ut}\|_0 + \|f^n - f\|_0 + (t_n - t)\Big(\|\partial_t r_1^n\|_0 + \|\Delta\partial_t U^n\|_0 + \|\nabla\partial_t P^n\|_0\Big)\Big]\|\xi_{ut}\|_0.
\end{aligned}
$$

Applying Young inequality, one finds

$$
\begin{aligned}
& 2\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}(T_1^n + T_2^n + T_4^n)ds \\
\leq \; & 2\Big[\Big(\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}\|\eta_{ut}\|_0^2 ds\Big)^{1/2} + \Big(\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}\|f^n - f\|_0^2 ds\Big)^{1/2} \\
& + \Big(\sum_{n=1}^{m} k_n^3(\|\partial_t r_1^n\|_0^2 + \|\Delta\partial_t U^n\|_0^2 + \|\nabla\partial_t P^n\|_0^2)\Big)^{1/2}\Big]\cdot\Big(\int_0^{t_m}\|\xi_{ut}\|_0^2 ds\Big)^{1/2}.
\end{aligned}
$$

For the term T_3^n, by integration for ξ_{ut} from t_{n-1} to t_n and summation by parts, there is

$$
\begin{aligned}
& \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_{ut})ds = \sum_{n=1}^{m}\frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_u^n - \xi_u^{n-1}) \\
= \; & \Big((I_h^m - I)(\frac{1}{k_m}U^{m-1}), \xi_u^m\Big) + \sum_{n=2}^{m} k_n\Big(\partial_t(I_h^n - I)(\frac{1}{k_n}U^{n-1}), \xi_u^{n-1}\Big) - \Big((I_h^1 - I)(\frac{1}{k_1}U^0), \xi_u^0\Big).
\end{aligned}
$$

By Lemma 2.1 and (4.32), one deduces that

$$
\begin{aligned}
& \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_{ut})ds \\
= \; & \Big((I_h^m - I)(\frac{1}{k_m}U^{m-1}), \xi_u^m - I_h^m\xi_u^m\Big) - \Big((I_h^1 - I)(\frac{1}{k_1}U^0), \xi_u^0 - I_h^1\xi_u^0\Big) \\
& + \sum_{n=2}^{m} k_n\Big(\partial_t(I_h^n - I)(\frac{1}{k_n}U^{n-1}), \xi_u^{n-1} - \hat{I}_h^n\xi_u^{n-1}\Big) \\
\leq \; & C\Big(\|h_1(I_h^1 - I)(\frac{1}{k_1}U^0)\|_0\|\nabla\xi_u^0\|_0 + \|h_m(I_h^m - I)(\frac{1}{k_m}U^{m-1})\|_0\|\nabla\xi_u^m\|_0 \\
& + \sum_{n=2}^{m} k_n\|\hat{h}_n\partial_t(I_h^n - I)(\frac{1}{k_n}U^{n-1})\|_0\|\nabla\xi_u^{n-1}\|_0\Big)
\end{aligned}
$$

$$\leq \; C\Big[\|h_1(I_h^1 - I)(\frac{1}{k_1}U^0)\|_0 + \|h_m(I_h^m - I)(\frac{1}{k_m}U^{m-1})\|_0$$

$$+\sum_{n=2}^{m} k_n\|\hat{h}_n\partial_t(I_h^n - I)(\frac{1}{k_n}U^{n-1})\|_0\Big] \cdot \mathscr{F}(t_i).$$

Combining above inequalities with (4.31), (4.32) with replacing t_m by t_i, we complete the rest of proof (4.26).

Finally, thanks to the inf-sup condition (2.1) and using (4.24), we yield that

$$\|\xi_p\|_0 \leq \beta^{-1} \sup_{0\neq v\in X} \frac{|(\nabla\cdot v, \xi_p)|}{\|\nabla v\|_0}$$

$$\leq \; \beta^{-1} \sup_{0\neq v\in X} \frac{1}{\|\nabla v\|_0}\Big[|(\xi_{ut}, v)| + |(\nabla\xi_u, \nabla v)| + |(\eta_{ut}, v)| + |\frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v)|$$

$$+|(f^n - f, v)| + (t_n - t)\Big(|\partial_t r_1^n(v)| + |(\nabla\partial_t U^n, \nabla v)| + |(\nabla\cdot v, \partial_t P^n)|\Big)\Big]$$

$$\leq \; C\Big[\|\xi_{ut}\|_0 + \|\nabla\xi_u\|_0 + \|\eta_{ut}\|_0 + \|h_n k_n^{-1}(I - I_h^n)U^{n-1}\|_0 + \|f - f^n\|_0$$

$$+k_n\Big(\|h_n\partial_t r_1^n\|_0 + \|\nabla\partial_t U^n\|_0 + \|\partial_t P^n\|_0\Big)\Big]. \tag{4.33}$$

Combining the results (4.25), (4.26), (4.29) with (4.33), we finish the proof.

Since (4.12) is quite similar in form to (3.4), we can prove the error estimates similar to Lemma 3.3 following the proof of Lemma 3.3.

Using required estimates of η_u and η_p in Theorem 4.1, we obtain the final theorem of this section. We introduce some notations for the purpose. Let

$$\mathscr{E}_8^m = \sum_{E\in\mathcal{E}_h} h_E\|[\nabla U^m - P^m\cdot\mathbf{I}]\|_{0,E}^2 + \sum_{K\in\mathcal{T}_h}\|\nabla\cdot U^m\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|r_1^m\|_{0,K}^2,$$

$$\mathscr{E}_9^0 = \sum_{E\in\mathcal{E}_h} h_E^3\|[\nabla U^0 - P^0\cdot\mathbf{I}]\|_{0,E}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|\nabla\cdot U^0\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^4\|r_1^0\|_{0,K}^2,$$

$$\mathscr{E}_{10}^m = \sum_{E\in\mathcal{E}_h} h_E^3\|[\nabla U^m - P^m\cdot\mathbf{I}]\|_{0,E}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|\nabla\cdot U^m\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^4\|r_1^m\|_{0,K}^2,$$

$$\mathscr{E}_{11}^m = \sum_{E\in\mathcal{E}_h} h_E^3\|\partial_t[\nabla U^m - P^m\cdot\mathbf{I}]\|_{0,E}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|\partial_t\nabla\cdot U^m\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^4\|\partial_t r_1^m\|_{0,K}^2.$$

Theorem 4.2. Let Ω be a bound polygonal domain with a sufficiently smooth boundary $\partial\Omega$. Assume that (u,p) and (U,P) are the solutions of (2.2) and (4.2), respectively. Under the assumptions of (**A1**), there exists a constant C, for $m \in [1,N]$ such that

$$\|\nabla(U-u)\|_0 \le C\mathscr{E}_5^m + \Big(2\mathscr{E}_1^m + \mathscr{E}_6^m + \mathscr{E}_8^m + \sum_{n=1}^{m} k_n\mathscr{E}_{11}^m\Big)^{1/2},$$

$$\|U-u\|_0^2 \le \|e_u(0)\|_0^2 + 3\Big(\mathscr{E}_1^m + \mathscr{E}_2^m + \mathscr{E}_4^m + \sum_{n=1}^{m} k_n\mathscr{E}_{11}^m\Big) + \mathscr{E}_8^m + \mathscr{E}_9^0 + \mathscr{E}_{10}^m,$$

$$\int_0^{t_m} \|\nabla(U-u)\|_0^2 ds \le \|u-U\|_0^2 + \sum_{n=1}^{m} k_n\mathscr{E}_8^m.$$

$$\int_0^{t_m} \|P-p\|_0 ds \le \Big(\mathscr{E}_1^m + \mathscr{E}_2^m + \mathscr{E}_6^m + \mathscr{E}_9^0 + \sum_{n=1}^{m} k_n\mathscr{E}_{11}^m + k_n^{-1}\mathscr{E}_4^m\Big)^{1/2} + C\mathscr{E}_5^m.$$

$$\|P-p\|_0 \le \|e_{ut}(0)\|_0 + C\mathscr{E}_5^m + 2\Big(\mathscr{E}_1^m + \mathscr{E}_6^m + \mathscr{E}_9^0 + \mathscr{E}_{11}^m + \sum_{n=1}^{m} k_n\mathscr{E}_{11}^m\Big)^{1/2} + \|f-f^n\|_0$$

$$+\|h_n\frac{1}{k_n}(I-I_h^n)U^{n-1}\|_0 + k_n\Big(\|\hat{h}_n\partial_t r_1^n\|_0 + \|\nabla\partial_t U^n\|_0 + \|\partial_t P^n\|_0 + \|f_t^n - f_t\|_0\Big).$$

Remark 4.1. Following the duality arguments given in Subsection 3.1, we can easily establish a posteriori error estimates for the completely discrete scheme (4.2). Since the techniques of the proof follow from a combination of the arguments give in Subsection 3.1 and Theorem 4.1, we skip the related analysis for simplification.

5. Numerical examples

In this section, we provide some numerical results to verify the performance of the established posteriori error estimators. For adaptive computations, the errors $\|e_u\|_1 = \|u-u_h\|_1$, $\|e_p\|_0 = \|p-p_h\|_0$, the error estimator η and the number of triangles (NT) in $\mathcal{T}_h$ are output of the adaptive algorithm. The experimental convergence rates are given by

$$\alpha_{e_w} = \frac{2 * log[\|e_w(i)\|_0/\|e_w(j)\|_0]}{log[NT(i)/NT(j)]}, \qquad e_w\ takes\ e_u,\ e_p\ or\ \eta.$$

The effectiveness index is defined as a rate of a posteriori error bound and an approximate norm of the actual error, here we use $\|e_u\|_1/\eta$. For a good estimator, this quantity should be a constant, independent of the mesh size and the time step. Although our theoretical findings do not include a proof of efficiency, numerical experiments provide evidences of the efficiency of the estimators. In all numerical tests, we take the final time $T = 1$. For simplicity, we do not perform time adaptive and take the time step k_n equal to 10^{-1}, and all the constants C involved in error indicators equal to 1. Our algorithm can be described as follows:

Algorithm. Let $\mathcal{T}_0$ is a regular triangulation.

(i). Compute on the shape-regular partition $\mathcal{T}_0$ with $t_0 = 0$.

(ii). Use the time step Δt and U^0 to compute $\|e_u\|_1$, $\|e_p\|_0$ and η on $\mathcal{T}_0$.

(iii). Begin the time loop with obtained $\mathcal{T}_{n-1}, \widehat{\mathcal{T}}_n$, U^{n-1} and P^{n-1}

(1) Let the time $t_n = \min(t_{n-1} + \Delta t, T)$,

(2) Use $\widehat{\mathcal{T}}_n, U^{n-1}$ and P^{n-1} to compute U^n and η^n,

(3) Adapt mesh $\widehat{\mathcal{T}}_n$ to obtain $\mathcal{T}_n$, use U^{n-1} and P^{n-1} to obtain U^n and P^n,

(4) For the next iteration, denote $\widehat{\mathcal{T}}_{n+1}$=$\mathcal{T}_n$,

(iv) End the time loop and finish the computation.

5.1. An analytical solution

For this test, our aim is to verify the theoretical analysis which has been established in the previous section by setting the body force f is given by the following exact solution

$$u_1 = \pi \sin^2(\pi x)\sin(2\pi y)\cos(t),$$

$$u_2 = -\pi \sin(2\pi x)\sin(\pi y)^2 \cos(t),$$

$$p = 10\cos(\pi x)\cos(\pi y)\cos(t).$$

with $\Omega = [0, 1]^2$. We adopt the MINI element to seek the exact solution and backward Euler scheme is used for time discretization. Table 1 presents the errors of velocity and pressure and convergence of order with different numbers of triangles (NT) at time $T = 1$. As expect, we can see that as NT becomes larger, the errors become smaller and smaller. The effectiveness index approaches 0.14, which is a constant independent of NT and time steps k_n.

Table 1: Results obtained using space-time algorithm based on MINI element.

NT	$\|e_u\|_1$	$\|e_p\|_0$	η	$\|e_u\|_1/\eta$	α_{e_u}	α_{e_p}	α_η
4466	0.0449492	0.0376184	0.480231	0.0936			
17053	0.0240032	0.0167303	0.175162	0.1370	0.9365	1.2095	1.5055
22530	0.0212163	0.0137918	0.146534	0.1448	0.8862	1.3869	1.2814
38098	0.0173495	0.011226	0.107594	0.1612	0.7660	0.7837	1.1760

5.2. Lid-driven cavity problem

Lid-driven cavity problem is a popular benchmark problem for testing numerical schemes. In this test, the fluid is enclosed in a square domain $\Omega = (0,1) \times (0,1)$, with $u = (1,0)$ on the upper side and $u = 0$ on the other three sides.

We start form the initial mesh with $h = 0.2$, see Fig.1 (a), the corresponding the profiles of both velocity and pressure are presented in Fig.1(b)-(d). Note that the successive iteration of the adaptive strategies creates more triangles in the two upper corners of the cavity as time increases, see Figs.2-3 (a). The profiles of velocity and pressure level lines are also presented at different time with adaptive computations, see Figs.2-3 (b)-(d). As expected, the oscillations of pressure at values obtained are absented. Finally, in order to show the prominent features of our adaptive algorithm, we describe the velocity and pressure contours obtained in adaptive mesh and uniform mesh with nearly the same number of triangles, see Figures 3 and 4. From these figures, we can see that the solution using a posteriori error analysis gives a more accurate approximation to the exact solution in critical regions.

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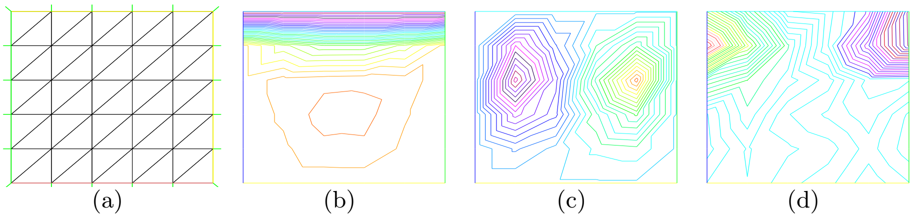

Figure 1: The mesh and the profiles of both pressure and velocity at t=0.001 with NT=50.

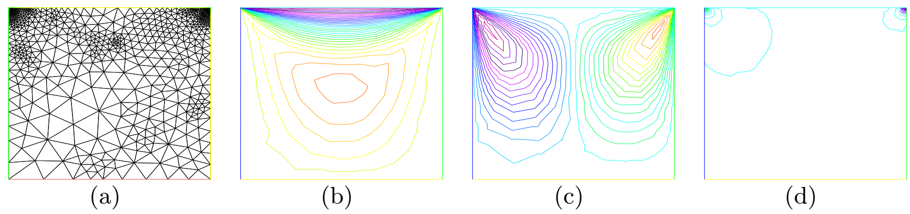

Figure 2: Adaptive mesh and the profiles of both pressure and velocity at t=0.5 with NT=804.

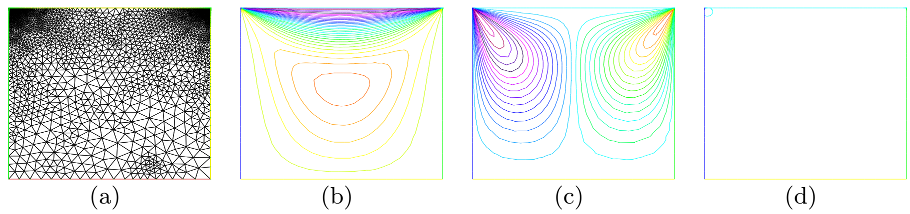

Figure 3: Adaptive mesh and the profiles of both pressure and velocity at t=1 with NT=8173.

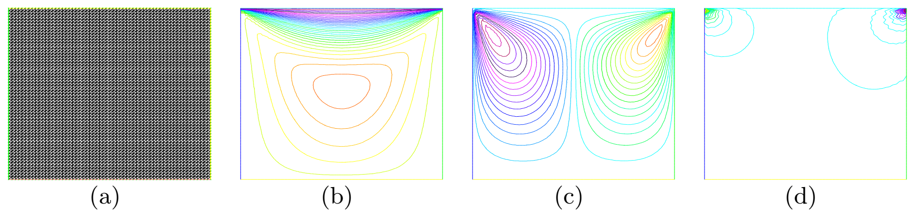

Figure 4: Uniform mesh and the profiles of both pressure and velocity at t=1 with NT=8192.