The quotient of two complex numbers is given in its polar form by the formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (r_2 \neq 0). \tag{4}$$

Since division is the inverse of multiplication, this formula can be established easily from formula (1). It follows, as a special case, that

$$\frac{1}{z} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)] = \frac{1}{r}(\cos\theta - i\sin\theta),$$

and, in view of equation (2), if z^{-n} is written for $1/z^n$,

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n}[\cos(-n\theta) + i\sin(-n\theta)] = \left(\frac{1}{z}\right)^n. \tag{5}$$

Thus the formula (2) and De Moivre's theorem (3) are valid when the exponent is any negative integer.

EXERCISES

1. Show that

(a) $\overline{\bar{z} + 3i} = z - 3i$; (b) $\overline{iz} = -i\bar{z}$;
(c) $\dfrac{(\overline{2+i})^2}{3-4i} = 1$; (d) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}\,|2z + 5|$.

2. Find one value of arg z when

(a) $z = \dfrac{z_1}{z_2}$ $(z_2 \neq 0)$; (b) $z = z_1{}^n$ $(n = 1, 2, \ldots)$;

(c) $z = \dfrac{-2}{1 + i\sqrt{3}}$; (d) $z = \dfrac{i}{-2-2i}$; (e) $z = (\sqrt{3} - i)^6$.

Ans. (a) arg z_1 − arg z_2; (b) n arg z_1; (c) $2\pi/3$; (e) π.

3. Use the polar form to show that

(a) $i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + 2i\sqrt{3}$; (b) $\dfrac{5i}{2+i} = 1 + 2i$;
(c) $(-1 + i)^7 = -8(1 + i)$;
(d) $(1 + i\sqrt{3})^{-10} = 2^{-11}(-1 + i\sqrt{3})$.

4. Let z_0 be a fixed complex number and R a positive constant. Show why point z lies on a circle of radius R with center at $-z_0$ when z satisfies

any one of the equations

(a) $|z + z_0| = R$; (b) $z + z_0 = R(\cos\phi + i\sin\phi)$,

where ϕ is real; (c) $z\bar{z} + \bar{z}_0 z + z_0\bar{z} + z_0\bar{z}_0 = R^2$.

5. Prove that (a) z is real if $\bar{z} = z$; (b) z is either real or pure imaginary if $z^2 = (\bar{z})^2$.

6. In Sec. 4, establish (a) formula (3); (b) formula (4).

7. Prove that (a) $\overline{z_1z_2z_3} = \bar{z}_1\bar{z}_2\bar{z}_3$; (b) $\overline{(z^4)} = (\bar{z})^4$.

8. Prove property (7), Sec. 5, on the absolute value of a quotient.

9. If $z_2z_3 \neq 0$, show that

(a) $\overline{\left(\dfrac{z_1}{z_2z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2\bar{z}_3}$; (b) $\left|\dfrac{z_1}{z_2z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

10. Give an algebraic proof of triangle inequality (9), Sec. 5.

11. If $|z_2| \neq |z_3|$, prove that

$$\left|\frac{z_1}{z_2 + z_3}\right| \leq \frac{|z_1|}{\big|\,|z_2| - |z_3|\,\big|}.$$

12. Prove that $|z|\sqrt{2} \geq |\mathcal{R}(z)| + |\mathcal{I}(z)|$.

13. Given that $z_1z_2 \neq 0$, use the polar form with arguments measured in radians to prove that

$$\mathcal{R}(z_1\bar{z}_2) = |z_1|\,|z_2|$$

if and only if arg z_2 = arg $z_1 \pm 2n\pi$ $(n = 0, 1, 2, \ldots)$.

14. Given that $z_1z_2 \neq 0$, use the result in Exercise 13 to prove that

$$|z_1 + z_2| = |z_1| + |z_2|$$

if and only if arg z_2 = arg $z_1 \pm 2n\pi$. Also, note the geometric verification of this statement.

15. Given that $z_1z_2 \neq 0$, use the result in Exercise 13 to prove that

$$|z_1 - z_2| = \big|\,|z_1| - |z_2|\,\big|$$

if and only if arg z_2 = arg $z_1 \pm 2n\pi$. Also, note the geometric verification of this statement.

16. Establish the formula

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1),$$

for the sum of a finite geometric series; then derive the formulas for example, contains all points of the plane except the origin and points on the positive x axis.

EXERCISES

1. Find all values of each of the following roots. Check graphically.

(a) $(2i)^{\frac{1}{2}}$; (b) $(-i)^{\frac{1}{3}}$; (c) $(-1)^{\frac{1}{3}}$; (d) $8^{\frac{1}{6}}$.

Ans. (a) $\pm(1 + i)$; (b) i, $(\pm \sqrt{3} - i)/2$;
(d) $\pm \sqrt{2}$, $(\pm 1 \pm i \sqrt{3})/\sqrt{2}$.

2. Find all values of

(a) $(-1 + i \sqrt{3})^{\frac{3}{2}}$; (b) $(-1)^{-\frac{3}{4}}$. *Ans.* (a) $\pm 2 \sqrt{2}$.

3. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.

Ans. $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

4. From the formula for the sum of a finite geometric series (Exercise 16, Sec. 7) show that, if w is any imaginary nth root of unity, then

$$1 + w + w^2 + \cdot \cdot \cdot + w^{n-1} = 0$$

5. Prove that the usual quadratic formula solves the quadratic equation $az^2 + bz + c = 0$ when the coefficients a, b, and c are complex numbers.

6. If m and n are positive integers, show (a) that

$$(z_1z_2)^m = z_1{}^m z_2{}^m;$$

(b) that the two sets of numbers $(z_1z_2)^{1/n}$ and $z_1{}^{1/n}z_2{}^{1/n}$ are the same; and hence (c) that the two sets $(z_1z_2)^{m/n}$ and $z_1{}^{m/n}z_2{}^{m/n}$ are the same.

7. Describe geometrically the region determined by each of the following conditions. Also, classify the region with the aid of the terms defined in Sec. 9.

(a) $|\mathcal{R}(z)| < 2$; (b) $|z - 4| > 3$; (c) $|z - 1 + 3i| \leqq 1$;
(d) $|\mathcal{I}(z)| > 1$; (e) $\mathcal{R}(z) > 0$; (f) $0 \leqq \arg z \leqq \pi/4$, $z \neq 0$.

Ans. (a), (b), (e) unbounded domain; (c) closed region, the closure of a bounded domain; (d) unbounded open region, not connected.

8. Describe each of these regions geometrically:

(a) $-\pi < \arg z < \pi$, $|z| > 2$; (b) $1 < |z - 2i| < 2$;
(c) $|2z + 3| > 4$; (d) $\mathcal{I}(z^2) > 0$;
(e) $\mathcal{R}\left(\frac{1}{z}\right) < \frac{1}{2}$; (f) $|z - 4| > |z|$.

CHAPTER 2

ANALYTIC FUNCTIONS

10. Functions of a Complex Variable. When z denotes any one of the numbers of a set S of complex numbers, we call z a complex variable. If for each value of z in S the value of a second complex variable w is prescribed, then w is a *function* of the complex variable z on the set S:

$$w = f(z).$$

The set S is usually some domain. Then it is called a *domain of definition* of the function w. The totality of values $f(z)$ corresponding to all z in S constitute another set R of complex numbers, known as the *range* of the function w.

A function is single-valued on a set S if it has just one value corresponding to each value of z in S. Let us agree that the term *function signifies a single-valued function* unless the contrary is clearly indicated. Most of our work with multiple-valued functions, such as $z^{\frac{1}{2}}$, can be carried out conveniently by dealing with single-valued functions, each of which takes on just one of the multiple values for each value of z in a specified domain.

The domain of definition of each of the functions

$$f_1(z) = z^3 + 2iz - 3, \quad f_2(z) = |z|, \quad f_3(z) = \frac{1}{z^2 + 1}$$

is the entire complex plane, except that f_3 is undefined at the two points $z = \pm i$. Note that f_2 is a real-valued function of the complex variable z; in fact its range is the nonnegative half of the real axis.

The functions $x = \mathcal{R}(z)$ and $y = \mathcal{I}(z)$ are also real-valued. If u and v are any two real-valued functions of the two real variables x and y, then $u + iv$ is a function of z. On the other hand, each given function $f(z)$ has specific real and imaginary