Again, let u and v satisfy all the hypotheses stated in Theorem 1; but now we assume that $z_0 \neq 0$. Under the coordinate transformation

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{5}$$

we can show by the chain rule for differentiation that, since the Cauchy-Riemann conditions (5), Sec. 17, are satisfied at z_0, then the conditions

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \qquad (r \neq 0) \tag{6}$$

are satisfied at that point. Here θ is measured in radians. Conversely, conditions (6) imply conditions (5), Sec. 17. Details are left to Exercise 7, below.

Equations (6) are the *Cauchy-Riemann conditions in polar coordinates.* They are useful in connection with the following alternate form of Theorem 1.

Theorem 2. *Let $u(r,\theta)$ and $v(r,\theta)$ each have a single real value at each point z in some neighborhood of a point (r_0,θ_0), and let u, v, and their partial derivatives of the first order with respect to r and θ be continuous functions of z at (r_0,θ_0) and satisfy the Cauchy-Riemann conditions (6) in polar coordinates, at that point, where $r_0 \neq 0$. Then the derivative $f'(z_0)$ of the function $f = u + iv$ exists, where $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$ and $z = r(\cos\theta + i\sin\theta)$; moreover, at the point $z = z_0$,*

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right). \tag{7}$$

The method used above to prove Theorem 1 can be used again here. The details are not so simple in this case, because of the nature of the formula for Δz in terms of Δr and $\Delta\theta$. The proof is outlined in Exercises 8 to 10 below.

EXERCISES

1. From the theorem in Sec. 17 show that $f'(z)$ does not exist at any point if $f(z)$ is

(*a*) $\bar{z}$; (*b*) $z - \bar{z}$; (*c*) $2x + xy^2i$; (*d*) $e^x(\cos y - i\sin y)$.

2. Use Theorem 1 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and use formula (2) to find $f'(z)$ and $f''(z)$, when

(*a*) $f(z) = iz + 2$; (*b*) $f(z) = e^{-x}(\cos y - i\sin y)$;
(*c*) $f(z) = z^3$; (*d*) $f(z) = \cos x\cosh y - i\sin x\sinh y$.

Ans. (*b*) $f'(z) = -f(z)$, $f''(z) = f(z)$; (*d*) $f''(z) = -f(z)$.

3. From the results given in Secs. 17 and 18 determine where $f'(z)$ exists and find its value there, when

(*a*) $f(z) = \frac{1}{z}$; (*b*) $f(z) = x^2 + iy^2$; (*c*) $f(z) = z\,\mathscr{I}(z)$.

Ans. (*a*) $f'(z) = -\frac{1}{z^2}$ $(z \neq 0)$; (*b*) $f'(x + ix) = 2x$; (*c*) $f'(0) = 0$.

4. If $f(z) = z^{\frac{1}{2}}$ where

$$z^{\frac{1}{2}} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad (r > 0,\ 0 < \theta < 2\pi),$$

use Theorem 2 to show that $f'(z)$ exists everywhere except along the positive real axis and at the origin and that $f'(z) = 1/[2f(z)]$.

5. If $f(z) = x^3 - i(y - 1)^3$, then $\partial u/\partial x + i\,\partial v/\partial x = 3x^2$. Why does $3x^2$ represent $f'(z)$ only at the point $z = i$?

6. The hypothesis in Sec. 17, that $f'(z_0) = a + ib$, can be stated as the condition that for each positive number ϵ there is a number δ such that

$$\left|\frac{\Delta f}{\Delta z} - a - ib\right| < \epsilon \quad \text{whenever } 0 < |\Delta z| < \delta.$$

Use that condition to derive equations (3) and (4), Sec. 17.

7. Under the coordinate transformations (5) and the continuity conditions stated in Theorem 1, obtain the partial derivatives of u and v with respect to r and θ in terms of derivatives with respect to x and y; then prove that, at the point z_0 $(z_0 \neq 0)$, conditions (6) are satisfied when conditions (5), Sec. 17, are satisfied, and conversely.

8. To simplify the formulas here, write $E(\theta) = \cos\theta + i\sin\theta$; then the polar form of z is $z = rE(\theta)$. If $\Delta z = (r_0 + \Delta r)E(\theta_0 + \Delta\theta) - r_0E(\theta_0)$, where $r_0 > 0$, derive the formulas

$$\begin{aligned}\Delta z &= E(\theta_0 + \Delta\theta)[\Delta r + ir_0\sin\Delta\theta + r_0(1 - \cos\Delta\theta)]\\ &= E(\theta_0 + \Delta\theta)[\Delta r + ir_0\,\Delta\theta + r_0\,\Delta\theta\, h(\Delta\theta)],\end{aligned}$$

where $h(\Delta\theta) = \dfrac{1 - \cos\Delta\theta}{\Delta\theta} - i\dfrac{\Delta\theta - \sin\Delta\theta}{\Delta\theta}$ and $\lim\limits_{\Delta\theta\to 0} h(\Delta\theta) = 0$.

9. When $r_0 > 0$ in Exercise 8, prove that $|\Delta\theta/\Delta z|$ is bounded for all Δr and $\Delta\theta$ when $|\Delta\theta|$ is sufficiently small; also, write $|\Delta z|^2$ in terms of Δr and $\Delta\theta$ and prove that $|\Delta r/\Delta z|$ is bounded when $|\Delta r| < r_0$.

10. Prove Theorem 2 by first deriving, with the aid of conditions (6) and results found in Exercises 8 and 9, these formulas:

$$\Delta f = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\Delta r + ir_0\,\Delta\theta) + \sigma_1\,\Delta r + \sigma_2\,\Delta\theta$$
$$= E(-\theta_0 - \Delta\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\Delta z + \sigma_1\,\Delta r + \sigma_3\,\Delta\theta,$$

where $\partial u/\partial r$ and $\partial v/\partial r$ are evaluated at z_0 and where

$$\sigma_n \to 0 \quad \text{as } \Delta z \to 0 \qquad (n = 1, 2, 3).$$

19. Analytic Functions. A function f of the complex variable z is *analytic at a point* z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in some neighborhood of z_0. It is *analytic in a domain* of the z plane if it is analytic at every point in that domain. The terms "regular" and "holomorphic" are sometimes introduced to denote analyticity in domains of certain classes.

The function $|z|^2$, for instance, is not analytic at any point, since its derivative exists only at the point $z = 0$, not throughout any neighborhood.

An *entire* function is one that is analytic at every point of the z plane, that is, throughout the entire plane. We have shown (Exercise 1, Sec. 16) that the derivative of every polynomial in z exists at every point; hence *every polynomial*

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \qquad (n = 0, 1, 2, \ldots)$$

is an entire function.

If a function is analytic at some point in every neighborhood of a point z_0 except at z_0 itself, then z_0 is called a *singular point*, or a *singularity*, of the function.

For example, we have seen that, if

$$f(z) = \frac{1}{z}, \qquad \text{then} \qquad f'(z) = -\frac{1}{z^2} \qquad (z \neq 0).$$

Thus f is analytic at every point except the point $z = 0$, where it is not continuous, so that $f'(0)$ cannot exist. The point $z = 0$ is a singular point. On the other hand, our definition assigns no singular points at all to the function $|z|^2$, since the function is nowhere analytic.

A necessary, but by no means sufficient, condition for a function to be analytic in a domain D is clearly that the function be continuous throughout D. The Cauchy-Riemann conditions are also necessary, but not sufficient. Two sets of sufficient conditions for analyticity in D are given by Theorems 1 and 2, Sec. 18, if the hypotheses stated in those theorems are satisfied at every point of D. But other useful sets of sufficient conditions arise in the following way from the conditions of validity of the differentiation formulas (Sec. 16).

The derivatives of the sum and product of two functions exist wherever the functions themselves have derivatives. Thus, *if two functions are analytic* in a domain D, *their sum and their product are both analytic* in D. Similarly, *their quotient is analytic* in D provided that the function in the denominator does not vanish at any point of D. In particular, the quotient P/Q of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

Let g be an analytic function of z in a domain D_1, and let R denote the range of $g(z)$ for all z in D_1. Then, if f is analytic in a domain D_2 that contains R, it follows from the conditions of validity of differentiation formula (6), Sec. 16, that the composite function $f[g(z)]$ is analytic in D_1. In brief, *an analytic function of an analytic function is analytic.*

As an illustration, the function $g(z) = 1 + z^2$ is entire. According to Exercise 4, Sec. 18, the function

$$f(z) = z^{\frac{1}{2}} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in its domain of definition. In particular, it is analytic in the upper half plane $\mathscr{I}(z) > 0$, an example of the domain called D_2 in the preceding paragraph. Since $\mathscr{I}[g(z)] = 2xy$, the range of g is confined to that half plane if $xy > 0$. Thus the composite function

$$f[g(z)] = (1 + z^2)^{\frac{1}{2}} \qquad (x > 0,\ y > 0)$$

is analytic in the domain D_1 consisting of the quadrant $x > 0$, $y > 0$ of the z plane.

We note also that *an entire function of an entire function is again entire.*

20. Harmonic Functions. Let the function $f = u + iv$ be analytic in some domain of the z plane. Then at every point