As a special case of the expansion (1), for example, when z is real, the representation

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

is valid for every real x.

By substituting Z^2 for z in the expansion (6), we note that

$$\frac{1}{1+Z^2} = \sum_{n=0}^{\infty} (-1)^n Z^{2n} \qquad \text{when } |Z| < 1,$$

since $|Z^2| < 1$ when $|Z| < 1$. When we make the substitution $z = -\alpha$, expansion (6) gives the sum of the infinite geometric series with α as the common ratio of adjacent terms; that is,

$$(7) \quad 1 + \alpha + \alpha^2 + \cdots + \alpha^n + \cdots = \frac{1}{1-\alpha} \quad \text{when } |\alpha| < 1.$$

The derivatives of the function $f(z) = z^{-1}$ are

$$f^{(n)}(z) = (-1)^n n!\, z^{-n-1} \qquad (n = 1, 2, \ldots ; z \neq 0),$$

and therefore $f^{(n)}(1) = (-1)^n n!$. Hence the expansion of this function by Taylor's series about the point $z = 1$ is

$$(8) \qquad \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n.$$

This expansion is valid when $|z - 1| < 1$, since the function is analytic at all points except $z = 0$.

As another example, let us expand the function

$$f(z) = \frac{1+2z}{z^2+z^3} = \frac{1}{z^2}\left(2 - \frac{1}{1+z}\right)$$

in a series of positive and negative powers of z. We cannot apply Maclaurin's series to f itself, since this function is not analytic at $z = 0$; but we can apply it to the function $1/(1 + z)$. Thus, when $0 < |z| < 1$, it is true that

$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2}(2 - 1 + z - z^2 + z^3 - \cdots)$$
$$= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \cdots .$$

EXERCISES

1. Show that, for every finite value of z,

$$e^z = e + e \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}.$$

2. Show that

$$(a) \quad \frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n \qquad \text{when } |z+1| < 1;$$

$$(b) \quad \frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n \qquad \text{when } |z-2| < 2.$$

3. Expand $\cos z$ by Taylor's series about the point $z = \pi/2$.

4. Expand $\sinh z$ by Taylor's series about the point $z = \pi i$.

5. Within what circle does the Maclaurin series for the function $\tanh z$ converge to the function? Write the first few terms of that series.

6. Prove that, when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}.$$

7. Make the substitution $z + 1 = Z$ in the Maclaurin series expansion (6) to obtain a representation of the function Z^{-1} in powers of $Z - 1$ that is valid when $|Z - 1| < 1$. Show that your result agrees with the Taylor series expansion (8).

8. Substitute Z^{-1} for z in expansion (6) and in its condition of validity to obtain an expansion of the function $(1 + Z)^{-1}$ in negative powers of Z that is valid everywhere outside the circle $|Z| = 1$.

Ans. $(1+Z)^{-1} = \sum_{n=0}^{\infty} (-1)^n Z^{-n-1} \quad (|Z| > 1)$.

9. Prove that when $x \neq 0$,

$$\frac{\sin(x^2)}{x^4} = \frac{1}{x^2} - \frac{x^2}{3!} + \frac{x^6}{5!} - \frac{x^{10}}{7!} + \cdots .$$

10. Represent the function

$$f(z) = \frac{z}{(z-1)(z-3)}$$

by a series of positive and negative powers of $(z - 1)$ which converges to $f(z)$ when $0 < |z - 1| < 2$.

$$Ans.\ f(z) = \frac{-1}{2(z-1)} - 3\sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}}.$$

58. Laurent's Series. Let z' denote any point on either of two concentric circles C_1 and C_2,

$$|z' - z_0| = r_1, \qquad |z' - z_0| = r_2,$$

about a point z_0, where $r_2 < r_1$ (Fig. 40). We shall prove the following theorem.

Theorem. *If f is analytic on C_1 and C_2 and throughout the region between those two circles, then at each point z between them $f(z)$ is represented by a convergent series of positive and negative powers of $(z - z_0)$,*

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$(2) \qquad a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots),$$

$$(3) \qquad b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z')\,dz'}{(z' - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots),$$

each integral being taken counterclockwise.

The series here is called *Laurent's series.*

In case f is analytic at every point on and inside C_1 except the point z_0 itself, the radius r_2 may be taken arbitrarily small. The expansion (1) is then valid when

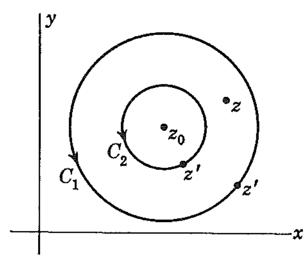

FIG. 40

$$0 < |z - z_0| < r_1.$$

If f is analytic at all points on and inside C_1, the integrand of the integral (3) is an analytic function of z' inside and on C_2, because $-n + 1 \leqq 0$; the integral therefore has the value zero and the series becomes Taylor's series.

Since the integrands of the integrals in formulas (2) and (3) are analytic functions of z' throughout the annular region, any closed contour C around the annulus can be used as the path of integration in place of the circular paths C_1 and C_2. Thus Laurent's series can be written

$$(4) \qquad f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n \qquad (r_2 < |z - z_0| < r_1),$$

where

$$(5) \qquad A_n = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

In particular cases, of course, some of the coefficients may be zero. In fact, the function

$$f(z) = \frac{1}{(z-1)^2},$$

for example, already has the form (4), where $z_0 = 1$. Here $A_{-2} = 1$ and all other A_n's are zero, which is in agreement with formula (5). Since this function is analytic everywhere except at the point $z = 1$, the curve C can be any closed contour enclosing that point.

The coefficients are usually found by other means than the use of the above formulas. For example, when $|z| > 0$, the expansions

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

and

$$e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!z^n}$$

follow from Maclaurin's series. We shall see (Sec. 62) that such representations are unique, so that these must be the Laurent series when $z_0 = 0$.

To prove the theorem we first note that, according to Cauchy's integral formula,

$$(6) \qquad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')\,dz'}{z' - z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')\,dz'}{z' - z},$$

since C_1 and C_2 form the boundary of a closed region throughout which f is analytic. In the first integral, as in the above proof of Taylor's theorem, we write

$$\frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)} = \frac{1}{z' - z_0} + \frac{z - z_0}{(z' - z_0)^2} + \cdots + \frac{(z - z_0)^{n-1}}{(z' - z_0)^n} + \frac{(z - z_0)^n}{(z' - z_0)^n(z' - z)}.$$