converges to $f(z)$ if $z \neq 0$. But series (4) clearly converges to $f(0)$ when $z = 0$. Thus $f(z)$ is represented by the convergent power series (4) for all z, and so f is an entire function. In particular, f is continuous at $z = 0$ and, since $z^{-1} \sin z = f(z)$ when $z \neq 0$, then

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} f(z) = f(0) = 1, \tag{5}$$

a result known beforehand because the limit here is the definition of the derivative of $\sin z$ at $z = 0$.

EXERCISES

1. By differentiating Maclaurin's series for $(1 - z)^{-1}$, obtain the representations

$$\frac{1}{(1 - z)^2} = \sum_{n=1}^{\infty} nz^{n-1}, \qquad \frac{2}{(1 - z)^3} = \sum_{n=2}^{\infty} n(n - 1)z^{n-2} \qquad (|z| < 1).$$

2. Expand the function z^{-1} in powers of $z - 1$; then obtain by differentiation the expansion of z^{-2} in powers of $z - 1$. Give the region of validity.

3. Integrate Maclaurin's series for $(1 + z')^{-1}$ along a contour interior to the circle of convergence from $z' = 0$ to $z' = z$ to obtain the representation

$$\text{Log}\,(z + 1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \qquad (|z| < 1).$$

4. If $f(z) = (e^{cz} - 1)/z$ when $z \neq 0$ and $f(0) = c$, prove that f is entire.

5. Expand $\sinh z$ in powers of $z - \pi i$ to prove that

$$\lim_{z \to \pi i} \frac{\sinh z}{z - \pi i} = -1.$$

6. If $f(z) = z^{-1}\,\text{Log}\,(z + 1)$ when $z \neq 0$ and $f(0) = 1$, prove that f is analytic throughout the domain $|z| < 1$.

7. If $f(z) = (z^2 - \pi^2/4)^{-1} \cos z$ when $z^2 \neq \pi^2/4$ and $f(\pm\pi/2) = -1/\pi$, prove that f is an entire function.

8. If a function f is analytic at z_0 and $f(z_0) = 0$, use series to show that

$$\lim_{z \to z_0} \frac{f(z)}{z - z_0} = f'(z_0).$$

Also note that this follows directly from the definition of $f'(z_0)$.

9. If f and g are analytic at z_0 and $f(z_0) = g(z_0) = 0$ while $g'(z_0) \neq 0$, prove that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

10. If f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, prove that the function g is analytic at z_0, where

$$g(z) = \frac{f(z)}{(z - z_0)^{m+1}} \qquad \text{if } z \neq z_0,$$

and

$$g(z_0) = \frac{f^{(m+1)}(z_0)}{(m + 1)!}.$$

62. Uniqueness of Representations by Power Series. The series in equation (3) of the preceding section is a power series that converges to $S'(z)$ everywhere within the circle of convergence C_0 of the series

$$\sum_{n=0}^{\infty} a_n z^n = S(z). \tag{1}$$

Consequently, that series can be differentiated term by term; that is,

$$S''(z) = \sum_{n=2}^{\infty} n(n - 1)a_n z^{n-2} \qquad (|z| < r_0).$$

Similarly, the derivative of $S(z)$ of any order can be found by successively differentiating the series term by term. Moreover,

$$S(0) = a_0, \qquad S'(0) = a_1, \qquad S''(0) = 2!a_2, \qquad \ldots,$$

so that the coefficients are those of the Maclaurin series expansion of $S(z)$,

$$a_n = \frac{S^{(n)}(0)}{n!}.$$

The generalization to series of positive powers of $(z - z_0)$ is immediate. Thus we have the following theorem on the uniqueness of the representation of functions in power series.

Theorem 1. *If the series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2}$$