$a_0 = 0$. When

(2) $$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

but $f^{(m)}(z_0) \neq 0$, then z_0 is called a *zero of order m* and

(3) $$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n}(z - z_0)^n \qquad (a_m \neq 0, |z - z_0| < r_0).$$

Let $g(z)$ denote the sum of the series in equation (3),

(4) $$g(z) = \sum_{n=0}^{\infty} a_{m+n}(z - z_0)^n \qquad (|z - z_0| < r_0).$$

Note that $g(z_0) = a_m \neq 0$. Since series (4) converges, g is a continuous function at z_0. For each positive number ϵ, therefore, a number δ exists such that

$$|g(z) - a_m| < \epsilon \qquad \text{whenever } |z - z_0| < \delta.$$

If $\epsilon = |a_m/2|$ and δ_1 is the corresponding value of δ,

$$|g(z) - a_m| < \tfrac{1}{2}|a_m| \qquad \text{when } |z - z_0| < \delta_1.$$

It follows that $g(z) \neq 0$ at any point in the neighborhood $|z - z_0| < \delta_1$, because if $g(z) = 0$ the inequality here is contradicted.

The argument remains valid if $m = 0$, in which case $g = f$ and $f(z_0) \neq 0$. Thus we have established the following theorem.

Theorem. *Unless a function is identically zero, about each point where the function is analytic there is a neighborhood throughout which the function has no zero, except possibly at the point itself. Thus the zeros of an analytic function are isolated.*

EXERCISES

1. Let g denote the function $\sin(z^2)$. Use Maclaurin's series (3), Sec. 62, for $g(z)$ to show that $g^{(2n-1)}(0) = 0$ and $g^{(4n)}(0) = 0$, where $n = 1, 2, \ldots$.

2. Use the expansion found in Example 4, Sec. 64, to show that, if C is the circle $|z| = 1$,

$$\int_C \frac{dz}{z^2 \sinh z} = -\frac{1}{3}\pi i.$$

3. Obtain the Maclaurin series representation

$$z \cosh(z^2) = z + \sum_{n=1}^{\infty} \frac{1}{(2n)!} z^{4n+1} \qquad (|z| < \infty).$$

4. Represent the function $(z + 1)/(z - 1)$ by (*a*) Maclaurin's series, and give the region of validity for the representation; (*b*) Laurent's series for the domain $|z| > 1$.

Ans. (*a*) $-1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; (*b*) $1 + 2\sum_{n=1}^{\infty} z^{-n} \quad (|z| > 1)$.

5. Obtain the expansion of the function $(z - 1)/z^2$ in (*a*) Taylor's series in powers of $z - 1$ and give the region of validity; (*b*) Laurent's series for the domain $|z - 1| > 1$.

Ans. (*a*) $\sum_{n=1}^{\infty} (-1)^{n+1} n(z - 1)^n \quad (|z - 1| < 1)$;

(*b*) $\sum_{n=1}^{\infty} (-1)^{n+1} n(z - 1)^{-n} \quad (|z - 1| > 1)$.

6. Obtain the Laurent series expansion

$$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{(2n + 1)!} z^{2n-1} \qquad (|z| > 0).$$

7. Give two Laurent series expansions, in powers of z, for the function

$$f(z) = \frac{1}{z^2(1 - z)}$$

and specify the regions in which those expansions are valid.

Ans. $\sum_{n=0}^{\infty} z^{n-2} \quad (0 < |z| < 1)$; $-\sum_{n=0}^{\infty} z^{-n-3} \quad (|z| > 1)$.

8. Write the two Laurent series in powers of z that represent the function $z^{-1}(1 + z^2)^{-1}$ in certain domains and specify those domains.

9. Obtain the first four terms of the Laurent series expansion

$$\frac{e^z}{z(z^2 + 1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1).$$

10. Obtain the first few terms of the Laurent series expansions

(a) $\csc z = \dfrac{1}{z} + \dfrac{1}{3!} z - \left[\dfrac{1}{5!} - \dfrac{1}{(3!)^2}\right] z^3 + \cdots \qquad (0 < |z| < \pi);$

(b) $\dfrac{1}{e^z - 1} = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{1}{12} z - \dfrac{1}{720} z^3 + \cdots \qquad (0 < |z| < 2\pi).$

11. Write the Laurent series expansion of the function $(z - k)^{-1}$ for the domain $|z| > |k|$, where k is real and $k^2 < 1$. Then write $z = \exp(i\theta)$ to obtain these formulas for sums of series of sines and cosines:

$$\sum_{n=1}^{\infty} k^n \sin(n\theta) = \frac{k \sin\theta}{p(k,\theta)}, \qquad \sum_{n=1}^{\infty} k^n \cos(n\theta) = \frac{k\cos\theta - k^2}{p(k,\theta)},$$

where $p(k,\theta) = 1 + k^2 - 2k\cos\theta$ and $k^2 < 1$.

12. Let $F(r,\theta)$ denote a function of z, where $z = r\exp(i\theta)$, that is analytic in some annulus about the origin that includes the circle $r = 1$. Take that circle as the curve C in the formula for the coefficients A_n in the Laurent expansion of $F(r,\theta)$ in powers of z, and show that

$$F(1,\theta) = \frac{1}{2\pi}\int_0^{2\pi} F(1,\theta')\,d\theta' + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} F(1,\theta')\cos[n(\theta - \theta')]\,d\theta'.$$

This is the Fourier series expansion of the complex-valued function $F(1,\theta)$ of the real variable θ on the unit circle. If $u(\theta)$ and $v(\theta)$ denote the real and imaginary components of $F(1,\theta)$, show that the above expansion is true when F is replaced everywhere by u or everywhere by v. The restrictions on the real-valued functions u and v here, however, are much more severe than they need be in order that those functions be represented by their Fourier series.[1]

[1] For other sufficient conditions, see, for instance, the author's book "Fourier Series and Boundary Value Problems," pp. 70, 86, 1941.

CHAPTER 7

RESIDUES AND POLES

66. Residues. If there is some neighborhood of a singular point z_0 of a function f throughout which f is analytic, except at the point itself, then z_0 is called an *isolated singular point* of f.

The function $1/z$ furnishes a simple example. It is analytic except at $z = 0$; hence the origin is an isolated singular point of that function. The function

$$\frac{z + 1}{z^3(z^2 + 1)}$$

has three isolated singular points, namely, $z = 0$ and $z = \pm i$.

As another example, the function

$$\frac{1}{\sin(\pi/z)}$$

has an infinite number of isolated singular points all lying on the segment of the real axis from $z = -1$ to $z = 1$, namely, $z = \pm 1$, $z = \pm\frac{1}{2}$, $z = \pm\frac{1}{3}$, etc. But the origin $z = 0$ is also a singular point; it is not isolated, since every neighborhood of the origin contains other singular points of the function.

Again, the function Log z has a singular point at the origin that is not isolated, because each neighborhood of the origin includes points on the negative real axis where Log z is not analytic.

When z_0 is an isolated singular point of f, a positive number r_1 exists such that the function is analytic at each point z for which $0 < |z - z_0| < r_1$. In that domain the function is represented by the Laurent series

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots,$$