On Local Solvability of Linear Partial Differential Equations - Part II

Fernando de Ávila Silva

Federal University of Paraná - Brazil

The main goal:

To present sufficient conditions for the local solvabillity of equation

$$Pu = f. \tag{1}$$

Remark

We recall that equation (1) *is said to be locally solvable, at a point* $x_0 \in \mathbb{R}^n$*, if there is a neighborhood* V *of* x_0 *such that for every function* $f \in C_c^\infty(V)$ *there is a distribution* u *in* V *satisfying* (1)

Remark

Here, P *is a linear partial differential operator of order* m*, with* ***smooth*** *coefficients. The leading symbol* $p(x, \xi)$ *is a homogeneous polynomial in* $\xi = (\xi_1, \ldots, \xi_n)$ *of degree* m*, where* $x = (x_1, \ldots, x_n)$*.*

Remark

Also, we are assuming that:

(a) *P is a principal type operator, namely,*

$$p(x_0, \xi_0) = 0, \ \ \textit{and} \ \ \xi_0 \neq 0 \implies \nabla_\xi p(x_0, \xi_0) \neq 0;$$

(b) *the real and imaginary parts of p are real analytic.*

Condition ¶

Definition

If $p(x,\xi) = A + iB$ and if $\nabla A \neq 0$ in a neighborhood of a point (x_0, ξ_0), the bicharacteristics of A are the oriented curves

$$\frac{dx}{ds} = \nabla_\xi A(x,\xi) \text{ and } \frac{d\xi}{ds} = -\nabla_x A(x,\xi)$$

The curves on which A vanishes are called the null-bicharacteristics of A

Condition ¶

On every null-bicharacteristics Γ of $\Re p$ the function $\Im p$ does not change sign, that is, we always have $\Im p \geq 0$ or $\Im p \leq 0$ on Γ.

The main results

The main results

Theorem (1)

Let P be a partial differential operator of principal type with analytic leading coefficients. If condition ¶ holds for x in a neighborhood of x_0, then x_0 has a neighborhood Ω_0 such that for every $f \in L^2(\Omega_0)$ there is a solution u of (1) in $H^{m-1}(\Omega_0)$.

Theorem (2)

In order that $Pu = f$ be locally solvable at every point, it is necessary and sufficient that condition ¶ hold. (P is partial differential operator of principal type with analytic leading coefficients).

Theorem (5)

Under the conditions of Theorem 1, assume that f belongs to H^k, k a positive integer; then there exists a neighborhood Ω_0^k of x_0 in which there is a solution u of (1) belonging to H^{k+m-1}.

Theorem (3)

Condition ¶ is equivalent to each of the following:

(a) *Every point x_0 has a neighborhood Ω_0 such that, for some constant $C > 0$,*

$$\|u\|_0 \leq C\|{}^tPu\|_{1-m} \text{ for all } u \in C_c^\infty(\Omega_0) \tag{3}$$

(b) *Every point x_0 has a neighborhood Ω_0 such that, for some constant $C > 0$,*

$$\|u\|_{m-1} \leq C\|{}^tPu\|_0 \text{ for all } u \in C_c^\infty(\Omega_0) \tag{4}$$

(c) *Given $\epsilon > 0$, any point x_0 has a neighborhood Ω_ϵ such that, for some constant $C > 0$,*

$$\|u\|_{m-1} \leq \epsilon\|{}^tPu\|_0 \text{ for all } u \in C_c^\infty(\Omega_\epsilon) \tag{5}$$

Furthermore, in any of these statements the operators P and ${}^tP$ may be interchanged or replaced by $p(x,D)$, the leading part of P.

Some remarks

Some remarks

- The essential point in the proofs of Theorems 1 and 3 is the proof that the condition ¶ implies condition c) of Theorem 3.
- By Bruno's seminars we know that if condition ¶ implies

$$\|u\|_{m-1} \leq \epsilon \|{}^tPu\|_0 \ \text{for all} \ u \in C_c^\infty(\Omega_\epsilon), \tag{5}$$

  then we obtain the proofs of Theorems 1, 3 and 5.
- By Alexandre's seminars we know that condition ¶ is invariant by a product of non vanishing functions.

Condition ¶ implies c)

The proof of this statement consists of three main steps:

(Step 1) In this step the authors reduce (5) to a similar estimate for a first order Ψ.D.O. satisfying ¶. Namely, in a neighborhood of a point (x_0, ξ_0) where p vanishes, assuming, say, $\partial p/\partial_{\xi_n} \neq 0$ there, we may factor

$$p = q(x,\xi) \cdot (\xi_n - \lambda(x, \xi_1, \ldots, \xi_{n-1})),$$

with $q \neq 0$ in the neighborhood. The problem is then reduced to one of an estimate of the form

$$\|u\|_0 \leq \epsilon \, \|Lu\|_0, \;\; \text{for} \;\; u \in C_c^\infty(\Omega_\epsilon), \qquad (7)$$

where

$$L = D_n - \lambda(x, D_1, \ldots, D_{n-1}).$$

(Step 2) This step consists in making a transformation to eliminate the real part a of $\lambda = a + ib$, tha is, reducing $\Re\lambda$ to ξ_n.

(Step 3) In this step the idea is to show (7) in case λ is pure imaginary.

We will show how to reduce the proof of (5) to (7)

We will show how to reduce the proof of (5) to (7)

...the proof is somewhat tedious...

Partition of unity on the sphere $|\xi| = 1$.

- Let $g_j(\xi), j = 1, \ldots, r$ be non-negative C^∞ function of ξ on $|\xi| = 1$, with $\sum g_j \equiv 1$. Extending g_j as a C^∞ function to all ξ-space which is homogeneous of degree zero for $|\xi| \geq 1/2$.

We will show how to reduce the proof of (5) to (7)

...the proof is somewhat tedious...

Partition of unity on the sphere $|\xi| = 1$.

- Let $g_j(\xi), j = 1, \ldots, r$ be non-negative C^∞ function of ξ on $|\xi| = 1$, with $\sum g_j \equiv 1$. Extending g_j as a C^∞ function to all ξ-space which is homogeneous of degree zero for $|\xi| \geq 1/2$.
- Consider the Ψ.D.O. (of order zero)

$$g_j(D)u(x) = (2\pi)^{-n} \int e^{ix\xi} g_j(\xi)\widehat{u}(\xi),\ u \in C_c^\infty(\mathbb{R}^n).$$

- We have $\sum g_j(D) \equiv I$ an infinitely smooth operator.

We will show how to reduce the proof of (5) to (7)

...the proof is somewhat tedious...

Partition of unity on the sphere $|\xi| = 1$.

- Let $g_j(\xi), j = 1, \ldots, r$ be non-negative C^∞ function of ξ on $|\xi| = 1$, with $\sum g_j \equiv 1$. Extending g_j as a C^∞ function to all ξ-space which is homogeneous of degree zero for $|\xi| \geq 1/2$.
- Consider the Ψ.D.O. (of order zero)

$$g_j(D)u(x) = (2\pi)^{-n} \int e^{ix\xi} g_j(\xi)\widehat{u}(\xi), \; u \in C_c^\infty(\mathbb{R}^n).$$

- We have $\sum g_j(D) \equiv I$ an infinitely smooth operator.
- For any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\|u\|_{m-1} \leq \sum \|g_j(D)u\|_{m-1} + C\|u\|_{m-2}. \tag{1.5}$$

- Locally, we may assume that the coefficients of p have compact support;

- Locally, we may assume that the coefficients of p have compact support;
- We may then consider the commutators

$$[p(x,D), g_j(D)]$$

which are Ψ.D.O.'s of order $\leq m-1$.

- It follows that

$$\sum \|p(x,D)g_j(D)u\|_0 \leq \|p(x,D)u\|_0 + C\|u\|_{m-1}. \tag{1.6}$$

- For now on we shall assume $x_0 = 0$.

The key point now is to show:

- the decomposition

$$p(x,\xi) = (\xi_n - \lambda(x,\xi')) \cdot q(x,\xi), \quad \text{(locally)}$$

with $q \neq 0$, and $\xi' = (\xi_1, \ldots, \xi_{n-1})$.

The key point now is to show:

- the decomposition

$$p(x,\xi) = (\xi_n - \lambda(x,\xi')) \cdot q(x,\xi), \quad \text{(locally)}$$

  with $q \neq 0$, and $\xi' = (\xi_1, \ldots, \xi_{n-1})$.
- that q and $\lambda(x,\xi')$ defines Ψ.D.O.'s.

From now on we will assume $x_0 = 0$. In particular, if $p(0, \xi_0) = 0$, then:

From now on we will assume $x_0 = 0$. In particular, if $p(0, \xi_0) = 0$, then:

- we may assume $|\xi_0| = 1$,

$$\nabla_\xi p(0, \xi_0) \neq 0 \text{ and } \frac{\partial p}{\partial_{\xi_n}}(0, \xi_0) \neq 0.$$

From now on we will assume $x_0 = 0$. In particular, if $p(0, \xi_0) = 0$, then:

- we may assume $|\xi_0| = 1$,

$$\nabla_\xi p(0, \xi_0) \neq 0 \ \text{ and } \ \frac{\partial p}{\partial_{\xi_n}}(0, \xi_0) \neq 0.$$

- we will denote $\xi' = (\xi_1, \ldots, \xi_{n-1})$.

- By the analyticity of p, there is an analytic function

$$\lambda(x, \xi') \quad \text{in} \quad X \times U',$$

where

From now on we will assume $x_0 = 0$. In particular, if $p(0, \xi_0) = 0$, then:

- we may assume $|\xi_0| = 1$,

$$\nabla_\xi p(0, \xi_0) \neq 0 \ \text{ and } \ \frac{\partial p}{\partial_{\xi_n}}(0, \xi_0) \neq 0.$$

- we will denote $\xi' = (\xi_1, \ldots, \xi_{n-1})$.

- By the analyticity of p, there is an analytic function

$$\lambda(x, \xi') \quad \text{in} \quad X \times U',$$

  where

  - X is a neigh. of the origin in $\mathbb{R}^n$
  - U' is a neigh. of ξ_0' in the $(n-1)$-dimensional space ξ'.

This function λ satisfies $\lambda(0, \xi') = \xi_{0,n}$ and

$$p(x, \xi) = (\xi_n - \lambda(x, \xi')) \cdot q(x, \xi)$$

where $q \neq 0$ in $X \times U$, where U is a neigh. of ξ_0.

This function λ satisfies $\lambda(0,\xi') = \xi_{0,n}$ and

$$p(x,\xi) = (\xi_n - \lambda(x,\xi')) \cdot q(x,\xi)$$

where $q \neq 0$ in $X \times U$, where U is a neigh. of ξ_0.

Remark

We point out that if the coefficients of p are merely C^∞, then there is such a C^∞ factorization since p is analytic in ξ.

This function λ satisfies $\lambda(0, \xi') = \xi_{0,n}$ and

$$p(x, \xi) = (\xi_n - \lambda(x, \xi')) \cdot q(x, \xi)$$

where $q \neq 0$ in $X \times U$, where U is a neigh. of ξ_0.

Remark

We point out that if the coefficients of p are merely C^∞, then there is such a C^∞ factorization since p is analytic in ξ.

Remark

By homogeneity, we see that $\xi_0' \neq 0$.

This function λ satisfies $\lambda(0,\xi') = \xi_{0,n}$ and

$$p(x,\xi) = (\xi_n - \lambda(x,\xi')) \cdot q(x,\xi)$$

where $q \neq 0$ in $X \times U$, where U is a neigh. of ξ_0.

Remark

We point out that if the coefficients of p are merely C^∞, then there is such a C^∞ factorization since p is analytic in ξ.

Remark

By homogeneity, we see that $\xi_0' \neq 0$.

Remark

Extending λ and q to conical neigh. Γ' (ξ'-space) and Γ (ξ-space), resp., for $|\xi| > 1/2$, we get

- *λ is homogeneous of degree 1.*
- *q is homogeneous of degree $m-1$.*
- *for $|\xi| < 1/2$ we them extend to be smooth.*

For X sufficiently small we can achieve that, for some $c_0 > 0$,

$$|q(x,\xi)| = c_0|\xi|^{m-1} \text{ in } X \times \Gamma$$

and

$$|p(x,\xi)| \geq c_0|\xi|^m \text{ in } X \times \Gamma_0$$

where Γ_0 is the cone in ξ space over U.