Distributions and Fourier Analysis
PPGM-UFPR

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For this class:

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- Basic notations;
- Support of a function;
- Test functions;
- Convolution.

Multi-index

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Remark

Given $\alpha, \beta \in \mathbb{Z}_+^n$ we say that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$, for each $j \in \{1, \ldots, n\}$. In particular, it is well defined the number

$$\binom{\alpha}{\beta} \doteq \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j}, \ \textit{for} \ \ \beta \leq \alpha.$$

Definition

Let $\Omega \subset$ be an open set and $k \in \mathbb{Z}_+$. We say that a function $f : \Omega \to \mathbb{C}$ belongs to $C^k(\Omega)$ if there exist

$$\partial^\alpha f \doteq (\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n})f, \ \forall \alpha \in \mathbb{Z}_+^n, \ \text{ such that } \ |\alpha| \leq k,$$

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- If $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$, we define

$$x^\alpha \doteq \prod_{j=1}^{n} x_j^{\alpha_j} = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$

Apllications

Taylor's formula

$$f(x+h) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha f(x) h^\alpha + k \int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha f(x+th) h^\alpha \, dt.$$

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Leibniz's rule

$$\partial^\alpha(f\cdot g) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} g.$$

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Definition (Test functions)

By $C_0^k(\Omega)$ we denote the space of all $u \in C^k(\Omega)$ with compact support. The elements of

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Remark

We may regard $C_0^k(\Omega)$ as a subspace of $C_0^k(\mathbb{R}^n)$.

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There exists a non-negative function $\phi \in C_0^\infty(\mathbb{R}^n)$ *with* $\phi(0) > 0$.

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Lemma (1)

Let $I \subset \mathbb{R}$ *be an open interval and* $a \in I$ *be a fixed point. Consider* $f : I \to \mathbb{R}$ *satisfying the following conditions:*

- f *is continuous;*
- f *is differentiable on* $I \setminus \{a\}$*;*
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Under these hypotheses f *is differentiable at* a *and* $f'(a) = 0$.

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Proof:

- Let $P(t)$ ($P(t) \neq 0$ if $|t| < 1$) be a polynomial with real coefficients and set

$$f(t) = \begin{cases} P(1/t)e^{-1/t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

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- it follows from Lemma (1) that f belongs to $C^1(\mathbb{R})$.
- The function $\phi(x) = f(1 - |x|^2)$ has the required properties.

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- *Let* $p \in \mathbb{R}^n$ *be fixed and* $\delta > 0$. *Then the function*

$$\phi_p(x) = \phi\left(\frac{x-p}{\delta}\right)$$

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    - $supp(\phi_p) \subset \{x \in \mathbb{R}^n;\ |x-p| \leq \delta\}$.

Definition

If $1 \leq p < \infty$ we define by $L^p_{loc}(\Omega)$ the spaces of all mensurable $f : \Omega \to \mathbb{C}$ such that for every compact set $K \subset \Omega$ we have $f|_K \in L^p_{loc}(K)$, namely,

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Exercise

Prove that if $f, g \in C(\Omega)$ *and*

$$\int_\Omega f(x)\varphi(x)dx = \int_\Omega g(x)\varphi(x)dx, \ \forall \varphi \in C_0^\infty(\Omega)$$

then $f = g$.

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If $f, g \in L^1_{loc}(\Omega)$ *and*

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Proof:

- We make use of the following:

Theorem (Lebesgue Differentiation Theorem)

If $h \in L^1_{loc}(\Omega)$ then

$$\lim_{t\to 0} \frac{1}{t} \int_{|x-y|<t} |h(x) - h(y)|dy = 0, \ a.e.$$

Definition (Convolution)

If u and v are in $C(\mathbb{R}^n)$ and either one has compact support, then the convolution product $u * v$ is the continuous function defined by

$$u * v(x) \doteq \int_{\mathbb{R}^n} u(x-y)v(y)dy, \ x \in \mathbb{R}^n.$$

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Theorem (Young's Inequality)

Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then the integral

$$f * g(x) \doteq \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

exists for almost everywhere $x \in \mathbb{R}^n$. Moreover, $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\| \leq \|f\|_1 \cdot \|g\|_p.$$