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Methods

On dimensions of ecological economics

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ABSTRACT

A recent paper (Ecological Economics 69, 2010, pp. 1604–1609) has addressed the issues of dimensional homogeneity of equations and non-linear transformations of variables in economic and ecological economic models. The authors argued that logarithmic transformation cannot be used when variables are dimensional, presented several examples of purportedly incorrect use in applied economics and ecological economics publications, and concluded that these applications "make no sense."

In this paper we show that this view goes against well established theory and practice of many disciplines including physics, statistics, biology, and economics, and rests on an inadequate understanding of dimensional homogeneity and the nature of empirical modeling in applied sciences. We believe that it is important to clarify that the use of dimensional variables in transcendental functions is in fact in accordance with the established scientific consensus so as to prevent further confusion from arising in ecological economics where addressing complex problems requires the synthesis of insights from many diverse disciplines to further our understanding of the environment–economy interface.

We also provide novel applications of dimensional methods to ecological economics and useful methodological references from several strands of scientific literature, not previously systematically consolidated, that should be of interest to every applied researcher.

1. Introduction

To further our understanding of the interdependence between the economy and the environment, ecological economics often seeks mathematical relationships among quantities that describe the phenomena under investigation. These quantities, variables and constants in our models, will often be *dimensional* in nature, i.e., their numerical value depends on the unit of measurement chosen. In a recent paper, published in this journal (Ecological Economics 69, 2010, pp. 1604–1609), Mayumi and Giampietro, hereinafter referred to as M&G, using arguments based on the Taylor's theorem of calculus, argued that exponential and logarithmic functions can only be applied to dimensionless numbers. They then reviewed several economics and ecological economics papers published over the past 50 years where this precept is purportedly not followed, resulting in applications that, according to the authors, "make no sense," and concluded that "it is unfortunate that many empirical and theoretical studies in economics, as well as in ecological economics, use dimensional numbers in exponential or logarithmic functions" and that "economists concerned with the biophysical and monetary aspects of ecological and economic interactions must understand the importance of *dimensional homogeneity*."

These are extraordinary claims that, if correct, would imply that most applications of statistics, economics, econometrics, and a considerable number of application in physics, which routinely employ logarithmic transformations of dimensional variables to model observed phenomena, simplify expressions, gain compliance with common statistical assumptions, estimate model parameters, and test hypotheses against observed data, among other things, are, using the authors' own words, "unacceptable." Relationships that capture the essence of ecological economics such as the stochastic IPAT (see, e.g., Dietz and Rosa, 1994; York et al., 2003) and the environmental Kutznets curve (see, e.g., Grossman and Krueger, 1993; Stern, 2010), where logarithms of dimensional variables are an essential part of the analysis, would also be unacceptable.

In this paper we show that the use of dimensional variables as arguments to transcendental functions in the examples criticized by M&G is in fact in accordance with the established scientific consensus. In the next section we review the concept of homogeneity of equations with physical quantities within traditional dimensional analysis and its extension to social sciences and economics. In Section 3 we apply dimensional analysis to economics and ecological economics problems and show how it can be useful in defining key variables, helping to construct models, and checking the "physical" validity of equations. In Section 4 we look at why often, logarithmic transformations of dimensional quantities that appear to be violate homogeneity, are actually part of a homogeneous expression. We also discuss, within an example from ecological economics, the role of dimensional constants. In

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Section 5 we look at nonhomogeneous models and empirical equations, their usefulness, and their correct interpretation. We show that, because of the complexity of the environment economic interactions, it is a misuse of dimensional analysis to insist that homogeneity rules must be rigidly and uncritically applied. We conclude with Section 6.

As M&G ignored the whole of the vast literature on dimensional analysis, we will try to amend this important omission, by providing useful references to a large body of literature scattered in various disciplines. The standard reference in dimensional analysis remains Bridgman (1931). Useful treatments of the topic include Langhaar (1951), Palacios (1964), de St. Q. Isaacson and de St. Q. Isaacson (1975), and Barenblatt (1996). For a historical account of the method see Macagno (1971) and Roche (1998). In economics, the earliest treatment is Jevons (1888). Several authors of economic books, acknowledging its importance, dedicate a chapter on dimensional analysis. They include Allen (1938), Shone (2002), and Neal and Shone (1976). The most authoritative exposition in the field of economics remains the book by De Jong (1967). Useful papers published on the subject in economics include De Jong and Kumar (1972) and Okishio (1982). Other, more specific references, will be provided in later sections.

2. Dimensional Variables and Their Homogeneous Equations

In order to avoid the mistakes in Mayumi and Giampietro (2010) and to apply dimensional methods to ecological economics correctly, we need to better understand the concept of dimensional homogeneity which has its roots in the fundamental theory of measurement in physics. Central to this understanding is the concept of "physical quantity." Note that in this context this terminology could be the source of some confusion, as in green accounting and ecological economics, "physical" is synonymous with "material" or "embodied energy" usually contrasted with monetary values as in Weisz and Duchin (2006). It is better to think of physical quantity as cardinally measurable property so that it becomes clear that it can include economic quantities such as "goods" and "money."

We need to address the following questions.

- What kind of numerical values representing physical properties can be considered a physical quantity, and
- what kind of restrictions apply to equations between physical quantities.

Most physical quantities have several units of measurement that are routinely employed in applications. We will use the expression *dimensional quantity*, to refer to a quantity whose numerical value depends on a specific unit of measurement. Following a well established convention, we will keep the concepts of dimensions and units distinct. A unit of physical quantities will be a standard for measurement of the same physical quantity, in the usual sense, like a meter or a kilogram. Dimensions can be regarded as generalized units. For example, anything that could be measured in mass units, such as kilograms, is considered to have the dimensions of mass, that we will denote with the symbol $[M]$.[1]

2.1. Base Quantities

Physical quantities are classified into two types: base quantities and derived quantities. The dimensions of basic quantities in classical mechanics are usually length $[L]$, mass $[M]$, and time $[T]$. The base quantities form a complete set of basic components for an open-ended system of "derived" quantities. This triplet LMT can be considered a *system of units* if it is sufficient to express all other quantities of interest in a specific field. It is important to stress that a given system of units is, to some extent arbitrary, and defined by convention. The base quantities are defined in entirely physical terms. Two physical quantities of the same kind, x_0 and x_1, are comparable if the following "ratio" is operationally and uniquely defined,

$$\frac{x_1}{x_0} \Leftrightarrow \lambda, \tag{1}$$

where λ is a numerical value indicating that x_1 is λ times greater than x_0.[2] This "axiom" is at the basis of the process by which physicists assign numbers to properties of objects. If we take the quantities x_0 to be a "standard unit," we say that the process of measurement produces the numerical value λ, the *number of measurement*. If we change units, say from x_0 to x_0', though the number of measurement will change, the quantity itself remains "physically" unaffected. Also, the ratio of any two samples of a base quantity remains constant when the base unit size is changed. In statistics and social sciences, quantities satisfying this property are said to be measured on a *ratio scale* (see, e.g., Hand, 2004; Stevens, 1946). When fundamental quantities of the same type are physically equated or added together, their corresponding numbers of measurement satisfy equations of the same form,

$$\underbrace{\begin{matrix} x_1 = x_2 \\ x_2 + x_3 = x_4 \end{matrix}}_{\substack{\text{Physical operations} \\ \text{(comparison and} \\ \text{concatenation) on} \\ \text{physical quantities} \\ \text{of the same type}}} \quad \begin{matrix} \Leftrightarrow \\ \Leftrightarrow \end{matrix} \quad \underbrace{\begin{matrix} \lambda_1 = \lambda_2 \\ \lambda_2 + \lambda_3 = \lambda_4 \end{matrix}}_{\substack{\text{Math operations} \\ \text{(equality and addi-} \\ \text{tion) on numbers of} \\ \text{measurement}}}$$

with x_i/x_0 giving λ_i, for $i = 1,\ldots,4$. Note that if the unit of measure is changed, so that x_i/x_0' produces λ_i', for $i = 1,\ldots,4$, the form of the equations remain unchanged: $\lambda_1' = \lambda_2'$ and $\lambda_2' + \lambda_3' = \lambda_4'$. If the above ratio is not defined, equalities might become inequalities by a simple change of units. In that case, the equation will be valid only for the particular choice of units. It is taken for granted that only quantities measured on a ratio scale are amenable to dimensional analysis (see, e.g., Bridgman, 1931; Krantz et al., 1971).

Outside basic physics, the choice of fundamental dimensions to adopt is far less clear and will depend on the area of application. For example, in macroeconomics, time $[T]$, money $[\$]$, goods $[R]$, and utility $[U]$, might be sufficient to derive all other quantities. See De Jong (1967) and Neal and Shone (1976) for a more detailed discussion. To apply dimensional analysis, we choose to treat many economic and social measurements as ratio scales, though they involve substantial pragmatic consideration. Many quantities that appear in ecological economics models are more appropriately measured on other scales, such as, following the well-known classification in Stevens (1946), the nominal, ordinal, and interval scales. As an example, though sums of money can be considered ratio scales, it does not follow that money, say, as a measure of utility in the exchange of goods, is also a ratio scale. In fact, research by Kahneman and Tversky (1979) has shown that zero is not an absolute reference point for monetary measurement, which would make the scale an interval one. There is a long debate in economics on cardinal and ordinal utility (see, e.g., Allen, 1956). Other variables used in ecological economics that are measured on the interval scale such as temperature (degrees Celsius, and Fahrenheit, but not with the Kelvin scale which has an "absolute" zero),[3] ordinal scale, such as "intelligence" in a growth model (as an example, IQ as in Morse, 2006), and on nominal scales, such as the political variables in a

[1] This square bracket notation dates back to Maxwell (see, e.g., Macagno, 1971) and will be used here to denote equivalence class as in Langhaar (1951) and De Jong (1967). Alternatively, it can be interpreted as a function meaning "dimension of."

[2] The division symbol here should be interpreted as a physical operation.

[3] Differences in temperature could be used instead.

relationship between environmental quality and income (for example, "signatory of an environmental treaty," "Democratic Country," and "Capitalist Country" as in Congleton, 1992), cannot be part of a dimensionally homogeneous equations as "ratios" are not invariant to changes in units.

2.2. Derived Quantities

Once a system of base quantities has been chosen, "derived" quantities can be introduced as necessary from these base quantities using definition linking the quantities involved.[4] For instance, in economics, the derived quantity of *price* is formed by "dividing" money [$] by goods [$R$]. It is important to keep in mind, that the operations of "division" and "multiplication" applied to physical quantities have no physical significance. No material entity is produced from dividing "money" with "goods." To symbolize this, we ignore the numerical values and, using the formal rules of multiplicative algebra, formulae involving physical quantities are transformed into expressions involving the dimensional symbols for these quantities. This allows as, to keep track of dimensions and units, with calculations involving physical quantities. For instance, price has dimension [MR^{-1}]. Bridgman (1931) argued that not all numbers obtained by inserting numerical values of base quantities into formulae can be considered *physical quantities*, only those that are ratio scales themselves can. For example, if we take LMT as our fundamental dimensions, it can be shown that only combination that take the power-law form [$M^{\mu}L^{\lambda}T^{\tau}$], where the exponents μ, λ, and τ are rational numbers, are allowed (see, e.g., Barenblatt, 1996; Bridgman, 1931; Langhaar, 1951; Palacios, 1964). A quantity is said to be *dimensionless* if all dimensional exponents are zero in which case we will say it has dimension [1]. No other functional form can represent "physical" quantities. This implies that transcendental functions, which cannot be expressed in terms of a finite sequence of basic algebraic operations, require dimensionless arguments to be part of a dimensionally homogeneous expression.

Not all "theoretically" valid derived dimensions obtained using this algebra are useful in practice. For example, if we take quantities of dimension mass [M] and length [L], whilst we cannot add them we can, for example, multiply them to obtain [ML], a measure of productivity in the transport sector such as the "ton-kilometer" units or divide them to obtain [ML^{-1}], a measure of linear density such as the "ton per kilometer" units. Random combination such as [$M^{-4}L^2$] or [M^5L^{-3}], though legitimate dimensions, have not yet been found useful. As we already seen, there is no physical concept for dividing mass by length. The fact the "ton per kilometer" has dimension [ML^{-1}] simply signifies that the numerical value of any measure defined in that way is increased by a factor ml^{-1} when the mass and length units are decreased by a factor m and l, respectively.

2.3. Physical Equations

Within physics, where the variables and constants are all assumed to be ratio scales, an equation is said to be *dimensionally homogeneous* if it is invariant to the choice of fundamental units of measurement. The study of these generalized homogeneous functions is the subject of *dimensional analysis* (see, e.g., Barenblatt, 1996; Birkhoff, 1960; Bridgman, 1931; Langhaar, 1951; Murphy, 1950; Palacios, 1964). The same definition is adopted in other disciplines such as statistics (see, e.g., Finney, 1977, p. 285), biology (see, e.g., Stahl, 1961, p. 359) and economics (see, e.g., De Jong, 1967, p. 28).

There are several useful consequences of this definition that can be used to check for dimensional homogeneity.

- Right hand side and left hand side of an equation must have the same dimensions,
- if sums appear in the equation its terms must have the same dimension.[5]

Equality of units and terms that appear in an equation can be used to check for homogeneity. Often, it is not readily apparent if an equation is dimensionally homogeneous. In such cases, the celebrated Π theorem of dimensional analysis of Buckingham (1914) provides an operational definition. The theorem loosely states that an unknown function[6] of several physical variables is dimensionally homogeneous if it can be rewritten in terms of a smaller set of dimensionless products of powers the original physical quantities. In physics, by postulating homogeneity of equations, this definition can be used to produce "laws" just from knowing the dimension of all variables involved in a relationship. In the next section, we will illustrate an application of this theorem to ecological economics in a simple setting where a strong physical understanding of the processes is available.

2.4. "Homogeneous" Equations with Non-physical Quantities

The idea of homogeneity for equations with scale ratio variables has been extended to more general numerical statements using a wider set of scales by the concept of meaningfulness developed within the Representational Theory of Measurement (RTM) which is the dominant theory of measurement in the social science (see, e.g., Luce et al., 1990). Roughly, a numerical statement[7] is considered *empirically meaningful* if it is invariant under legitimate scale transformations of the underlying variables (see, e.g., Adams et al., 1965; Suppes and Zinnes, 1963). We can apply the idea of meaningfulness to the logarithms of dimensional variables. Consider the following equation,

$$\log y = \alpha x. \tag{2}$$

At first, let us assume that variables are measured on ratio scales, for instance, with y denoting emission measured in units of mass and x denoting GDP, measured in currency units. Ratio scales can be expressed in different units by multiplying them by some numerical factor. If we change units, Eq. (2) becomes,

$$\log(ky) = \alpha(mx), \tag{3}$$

i.e.,

$$\log y = \hat{\alpha}x - \log k \tag{4}$$

with $\hat{\alpha} = m\alpha$. Eq. (4) cannot be reduced to the original form because of the $\log k$ term so that Eq. (2) is empirically "meaningless." Of course, this result agrees with dimensional analysis. On the other hand, if y is a variable unique up to a power transformation, i.e., measured on a so-called log interval scale such as utility (see, e.g., Stevens, 1957), then Eq. (2), after changing units, becomes,

$$\log\left(y^k\right) = \alpha(mx) \tag{5}$$

which reduces to

$$\log y = \hat{\alpha}x \tag{6}$$

[4] An algebraic approach to dimensional analysis, has been developed by, among others, Drobot (1953), Thun (1960), and Whitney (1968). As will be clear from later discussions, this approach is generally too narrow to be used in ecological economics models.

[5] See Langhaar (1951, p. 50) for an explanation of why using this as a definition of dimensional homogeneity is mathematically unsatisfactory.

[6] Depending on the proof the function needs to satisfy a number of differentiability conditions.

[7] It does not have to be an equation, but other mathematical statements including, say, inequalities.

with $\hat{\alpha} = \frac{m\alpha}{k}$. Thus, Eq. (2), in this context, is invariant under the admissible transformations and is therefore "meaningful." We can see from these examples that whether or not it is legitimate to use logarithms of dimensional variables depends not only on the measurement scale but also on the nature of the statement. For a more detailed discussion of meaningfulness see Suppes and Zinnes (1963) and Narens (1984). For a discussion within statistics see Hand (2004). For a recent collection of applications of RTM in economics see the book edited by Boumans (2007).

3. Illustrative Applications of Dimensional Analysis to Ecological Economics

If we assume all variables are physical quantities, dimensional analysis can be used to aid modeling mainly in order to,[8]

- check ex post for internal consistency of relationships, or
- ex post constrain functional relationship between physical quantities.

In economics and related disciplines, checking equations seems to be the most frequent use of dimensional analysis. However, in physics, dimensional analysis arguments are more often been used to arrive at physical laws without an a priori knowledge of the functional forms. Dimensional analysis can be a powerful tool in ecological economics. Just from basic principles, by choosing the appropriate physical quantities, we can appreciate the key elements driving a result without the need of a more formal analysis. As an example, dimensional analysis applied to the study of alternative energy sources can tell us, that in the case of wind turbines, if we double the size of the blades we might expect to quadruple the power output, whereas if the wind speed is doubled the power output is increased by a factor of 8 (see, e.g., Andrews and Jelley, 2007).

3.1. Production Function

For example, to apply this to a production function using strong physical grounding, let us consider an extremely simplified production function with one input and one output of a steel container. Input will be steel, x, and output the capacity of the container y. Since these are physical quantities, initially we can assume the monomial relationship between input and output,

$$y = A \cdot x^b \tag{7}$$

where A is an unknown constant. Since input is proportional to the area of the container, and output to the volume, x will have a dimension of $[L^2]$ and y of $[L^3]$. In terms of dimensions the equation is

$$\left[L^3\right] = \left[L^2\right]^b. \tag{8}$$

To balance the equation for homogeneity, it is required that $3 = 2b$, so that $b = {}^3/_2$. This gives

$$y = A \cdot x^{\frac{3}{2}}. \tag{9}$$

If we double inputs, output increases by a factor of 1.5, production increases more than the increase in inputs. We have managed to frame the classic argument to justify increasing returns for some productions using dimensional considerations. This application clearly illustrates the power and limitations of dimensional homogeneity arguments. We indeed obtained a functional relationship between physical quantities, however, to assume homogeneity a priori, we need to include all relevant variables, which requires a detailed understanding of the problem, and results need to be empirically verified. Failing to do so will produce nonsensical results (see, e.g., Barenblatt, 1996; Bridgman, 1931; Langhaar, 1951; Murphy, 1950; Palacios, 1964).

3.2. IPAT Identity

Consider the central equation in ecological economics, proposed by Commoner (1972) and Ehrlich and Holdren (1972), known as the IPAT equation, based on dimensional analysis arguments. IPAT is an identity stating that environmental impact (I) is the product of population (P), affluence (A), and technology (T), i.e., $I = P \cdot A \cdot T$. Waggoner and Ausubel (2002, p. 7860) stated that "Dimensions provide an ironclad audit of forces proposed for an index of impact, and the simplicity guarantees wide applicability." The identity helps also to define indicators of technology, and is an indispensable tool in the quantitative accounting of environmental/population/economic interactions (see, e.g., Chertow, 2000, p. 14). For dimensional checks purposes, let us consider using emissions $[E]$, population $[N]$, money $[\$]$, and time $[T]$ as our primary dimensions.[9] Impact I is a flow variable[10] with dimension of emissions per unit of time $[ET^{-1}]$, population size P is a stock variable[11] with dimension $[N]$, affluence A is also a flow variable, which can be interpreted as the value of goods and services produced over a given time period per capita, with dimension $[\$\, N^{-1}\, T^{-1}]$, and technology T can be interpreted as emissions per unit of output measured in monetary units, with dimension $[E\$^{-1}]$. The IPAT identity is, trivially, dimensionally homogeneous,

$$\left[ET^{-1}\right] = [N] \frac{\left[\$T^{-1}\right]}{[N]} \frac{\left[ET^{-1}\right]}{\left[\$T^{-1}\right]}, \tag{10}$$

that is, if we change the units in which the variables are measured, the form of the relationship remains unchanged. The simplicity of this model should be a reminder that dimensional homogeneity by itself can not guarantee its validity and usefulness.

3.3. Environmental Kuznets Curve

There is an important class of homogeneous equations that has many applications in understanding dimensional homogeneity in regression analysis and in empirical ecological economics models such as the environmental Kuznets curve (EKC). In general, a mathematical equation will be homogeneous if it is amenable to a geometric

[8] In practice, as modeling is an iterative process where empirical tests and observations are used to improve models, this distinction is less clearcut. The role of dimensional analysis can be far more critical in experimental sciences. For example, Gibbings writes that in physical sciences "Dimensional analysis is a powerful means in the design, the ordering, the performance and the analysis of experiment and also the synthesis of the resulting data" (Gibbings, 2011, p. 3).

[9] Quantities which are simply counted, such as the number of people or the number of occurrences of an event, are in general considered dimensionless (see, e.g., Taylor and Thompson, 2008 p. 12). Note that there are several well known cases of completely different physical quantities that are either dimensionless or have identical dimensions. In this case, it is meaningless to compare or add them. A classic example in physics is that of torque and energy that share the same dimension $[ML^2T^{-2}]$ (for other examples, see e.g., Szirtes, 2006, p. 42). In economics, both "interest rate" and "velocity of money" have dimensions $[T^{-1}]$ though they are clearly not additive (see, e.g., De Jong, 1967, pp. 16–17). The same is true for dimensionless quantities. It is not acceptable to add, say, angles to proportions. In fact, to make dimensional checks feasible we should attempt to assign dimensions to every variable in our equations. That is why here we use a population $[N]$ dimension (cf. Ewing, 1973).

[10] See, e.g., Common and Stagl (2005, pp. 88–104). In general, a *stock* is a quantity existing at a point in time. A stock quantity does not involve the time dimension $[T]$. A *flow*, on the other hand, is a quantity over a period of time. A flow variable must include the dimension $[T^{-1}]$ (see, e.g., Shone, 2002, pp. 8–12). The fact that quantities in the IPAT equation tend to be flows is typically ignored.

[11] Population can also be defined as a flow variable as its change over a defined period $[N\, T^{-1}]$ (see, e.g., Chertow, 2000, p. 16).

interpretation (see, e.g., de St. Q. Isaacson and de St. Q. Isaacson, 1975, pp. 40–42). Consider a quadratic function,

$$y = ax^2 + bx + c. \tag{11}$$

This function can be interpreted geometrically as the graph of a parabola in a euclidean space. In this case, the y and x coordinates, interpreted as distances from the horizontal and vertical axis, have both dimensions of length $[L]$, c represents the y-intercept, i.e., the value of y when $x=0$, which therefore has also dimension $[L]$, b is the slope at $x=0$ with dimension $[\mathrm{d}\,y/\mathrm{d}\,x]=[L/L]$, a is the second derivative of y with respect to x with dimension $[\mathrm{d}^2/\mathrm{d}\,x^2]=[L/L^2]=[L^{-1}]$. The equation is homogeneous, as is easily verified

$$[L] = \left[L^{-1}L^2\right] + [1L] + [L]. \tag{12}$$

The equation of a parabola could for example describe the trajectory of a thrown object if we measure height and distance in meters. M&G's statement that $4m^2+4m^3$ "does not make any sense" (p. 1608), left unqualified is wrong, as the authors ignore the likely presence of dimensional constants. In fact, if m represents a length quantity and the dimensions of the coefficients are [1] and $[L^{-1}]$ respectively, this equation could describe the vertical and horizontal location measured in meters of a cubic curve.[12]

This approach can be used to determine dimensions of the popular environmental Kuznets curve at the core of the relationship between environment and socio-economic activities. Considerer the following simple version in levels,[13]

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 \tag{13}$$

with y having dimension of emission per period $[ET^{-1}]$, and x of money per period $[\$T^{-1}]$. The dimensional analysis approach for more standard log-transformed version of the EKC will be discussed in Sections 4 and 5. Based on the geometric interpretation, β_0 has dimension of $[ET^{-1}]$, β_1 of $[E\$^{-1}]$ and β_2 of $[E\$^{-2}]$.

3.3.1. Problems with Arguments Based on the Taylor's Expansion

Central to M&G's analysis is what they call a re-visitation of exponential and logarithmic functions. They produce four Taylor expansions to justify the fact that arguments of transcendental functions have to be dimensionless. For example, M&G show the expansion of e^x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{14}$$

M&G write: "to make sense, each element $x/k!$ [should be $x^k/k!$] ($k=0,\ 1,\ldots$) must either have the same dimension or be a dimensionless pure number." Note that the terms, being increasing powers of x, cannot possibly have the same dimension, they can only be dimensionless. M&G then continue by concluding that "If neither of these two conditions are satisfied, the summation cannot be done." We will show that their arguments, as presented in the paper, are fundamentally flawed. In particular, they ignore the presence of dimensional constant in their analysis, and misunderstand the concept of derived dimensions.

3.4. Dimensional Constant

In their example, M&G ignored the impact of a dimensional quantity that they themselves introduced. To illustrate that it is "absurd to put a dimensional number into a logarithmic function" (p. 1605), M&G showed the expansion of $\log(1+x)$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{15}$$

and stated "Suppose now we assign to x the value of 0.5 U.S. dollars." Again they concluded the equation "would not make any sense." Notice that to add a quantity x measured in a monetary unit to the constant "1", it must be the case that the constant has also a dimension of money. The authors failed to incorporate this aspect into their analysis. If we rewrite the expansion making this explicit, considering $\log(1+x)$ as a special case of $\log(a+x)$, it becomes clear that dimensional homogeneity is maintained,

$$\log(a+x) = \log a + \log\left(1+\frac{x}{a}\right) = \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \cdots \tag{16}$$

in fact, for $a=1$, all terms are now dimensionless,[14] as dimensions cancel out.

There is a more general issue with the use of transcendental functions in applied fields. There are many instances in fields such as applied mathematics and statistics of transcendental functions applied to apparently dimensional arguments. Consider the continuous probability density $f(x) = \frac{1}{30}e^{-x/30}$, from the standard statistics book by Miller and Miller (2008, p. 101), where $x>0$ is a random variable measured in thousands of km, thus of dimension length $[L]$. Its density must have dimension $[L^{-1}]$, so that when multiplied by dx gives a dimensionless probability (see, e.g., Finney, 1977), together with the requirement that the argument of the exponential function has to be dimensionless, this implies that both constants 1/30 have to be of dimension $[L^{-1}]$. Consider the equation of a catenary $y = 10e^{x/20} + 10e^{-x/20}$ from an exercise in the calculus textbook by Stewart (2007, p. 260), where x and y are reportedly measured in meters. Again, we have to assume that constants are dimensional. The important point is that the argument of any transcendental function, such as e^{3x}, when x is dimensional, is made implicitly dimensionless by assuming that the dimension of the constant is the reciprocal of that of x. We need to keep in mind that the constant will change value if the units are changed. Taking logarithms of dimensional variables does not produce "meaningless" results but results that depend on the units of measurement used.

3.5. Derived Quantities

Another serious problem affecting M&G's analysis is their misunderstanding the basic concept of derived dimensions. They imply that using dimensional quantities in the nonlinear terms of the Taylor expansion is itself not permissible. To illustrate the "nonsense" of this operation they write (p. 1605),[15]

$$\begin{array}{ccccc} 1\ \text{US\$} & + & 1\ \text{US\$} & = & 2\ \text{US\$} \\ 1\ \text{US\$} & \times & 1\ \text{US\$} & = & 1\text{“square”US\$?} \\ 1\ \text{US\$} & \times & 1\text{“square”US\$?} & = & 1\text{“cubic”US\$?} \end{array} \tag{17}$$

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[12] In fact, even with dimensionless constants, $4m^2+4m^3$ could be dimensionally homogeneous. Consider the case where m is a 2×2 matrix with dimensions $\begin{pmatrix} [1] & [M] \\ [M^{-1}] & [1] \end{pmatrix}$, by the rule of matrix multiplication, the dimensions of m, m^2, m^3, and so on, are all the same. If this sounds like an artificial example, consider the case of input output analysis where the Leontief inverse can be defined as $(I-A)^{-1} = I + A + A^2 + A^3 + \cdots$ (see, e.g., Miller and Blair, 2009, pp. 31–34), where the technology matrix $(A)_{ij} = a_{ij}$ has dimension of good i over good j $[R_i R_j^{-1}]$. This shows that the elements of the Leontief inverse have the same dimension as elements of A.

[13] Omitting "per capita," to simplify notation. See also footnote 9.

[14] Note that "zero," $\log(1)=0$, is considered having no dimension in standard dimensional analysis axiomatization (see, e.g., Krantz et al., 1971, p. 461).

[15] They also present a very confusing and worrying "pictorial" representation of a Taylor expansion of an "exponential function with US$" that seems to imply that the authors think that mathematical functions can have real objects as arguments instead of real numbers.

The authors claim that "one dollar plus one dollar makes perfect sense, but one dollar times one dollar does not make any sense at all." The first part of the statement referring to a fundamental unit is self-evidently true, whereas the second part, on a derived unit, is absurd.[16] We have seen in Section 2 that even though operations, such as the multiplication and division of fundamental physical quantities, have in no sense physical meanings, they often produce useful quantities and relationships. For example, in the EKC, income squared, with dimension of money squared, is interpreted as a term that captures factors impacting the environment that vary with the level of income.

4. Hidden Homogeneity and Dimensional Constants

Dimensional homogeneity of equations with physical quantities can be masked by mathematical manipulation of logarithms. See, as authoritative examples, Ipsen (1960, pp. 118–119), Bridgman (1931, p. 73), Palacios (1964, p. 52), de St. Q. Isaacson and de St. Q. Isaacson (1975, p. 34), and Hand (2004, p. 210). In physical sciences there are many equation that appear to be nonhomogeneous such as the physical chemistry equation in Bridgman (1931, p. 74) and de St. Q. Isaacson and de St. Q. Isaacson (1975, p. 34). Another example is British physicist G.I. Taylor's famous scaling law (see, e.g., Barenblatt, 1996, p. 6).

As pointed out by de St. Q. Isaacson and de St. Q. Isaacson (1975, p. 34), if we consider a velocity defined as space s of dimension $[L]$ over time t of dimension $[T]$, i.e., $v=s/t$, the logged form $\log(v)=\log(s)-\log(t)$, though actually homogeneous by Buckingham's Π theorem, as it can be reduced to a function of a dimensionless product, $log\left(\frac{vt}{s}\right)=0$, would not normally be used in practice. For most applications in statistic and economics, the log transformation is instrumental in obtaining the parameter of the models for estimation and testing, the transformed variables are of no interest. There are many examples of statistical and econometric regression equations derived from deterministic dimensionally homogeneous models by allowing for stochastic components and using logarithms of dimensional variables in order to incorporate non-linearities, to estimate model parameter, and to quantify and test theoretical models. Worked out examples can be found in Seber and Lee (2003, p. 4), Wooldridge (2008, pp. 212–215), and Baltagi (2011, p. 80). The logarithmic form allows parameters to acquire the convenient interpretation of dimensionless elasticity (see, e.g., Wooldridge, 2008, pp. 212–215).

Consider, as an example in ecological economics, the more general version of the IPAT relationship, which leads to the stochastic IPAT, introduced by Dietz and Rosa (1994) and York et al. (2003),

$$I=aP^{\beta}A^{\gamma}T^{\delta}. \tag{18}$$

This and analogous modications of IPAT allow for less trivial and possibly more relevant indicators of Technology.[17] For this equation to be dimensionally homogeneous, the "scaling" constant a has to have dimension $[E^{1-\delta}N^{\gamma-\beta}\$^{\delta-\gamma}]$. Note that the presence of a dimensional constant allows the choice of an arbitrary unit for I.[18] The inclusion of dimensional constants raises a few important issues. The main problem is that it appears that every equation can be made dimensionally homogeneous with the introduction of an ad hoc dimensional constant. In physics, constants have often dimensions that can be deduced from fundamental principles, though it is generally accepted that not all dimensional constants can be explained that way (see, e.g., de St. Q. Isaacson and de St. Q. Isaacson, 1975, p. 36). This does not cause any particular problems as long as the dimensions of all quantities except for the constants are known (de St. Q. Isaacson and de St. Q. Isaacson, 1975, p. 36).[19]

Other disciplines, to make dimensional analysis operational, have to include dimensional constants that might be harder to justify a priori. We take the view that the "ad hoc" dimensions of the scale parameter reflect the self-evident truth that this model is a very crude approximation of reality. We should not be surprised that the coefficient's dimension reflects in part our ignorance of the processes involved and the limitations of the model. To distinguish equations that require the dimensions of the coefficients to be considered in checking for homogeneity from other homogeneous equations that more likely to be encountered in physics, Murphy (1950, p. 18) and Johnson (1972) suggested the expression *restricted homogeneous*.

This example also illustrates another important difference between equations that we are likely to encounter in ecological economics as opposed to physics. In practice, because none of the values for the "ecological elasticities" β, γ or δ are theoretically given for any specific IPAT relation, unlike analogous equations in physics, like Newton's gravitational equation, where exponents are given by the quantity's definitions, they must be estimated. Taking logs, Eq. (18) can be expressed in the form,

$$\log I = \log a + \beta \log P + \gamma \log A + \delta \log T \tag{19}$$

which can be fitted to data by conventional least squared methods, to obtain estimates of the parameters of the model. It is reasonable to expect a wide range of values that depend on the application, samples, definition of variables, method used, and so on. Variables in economics and related fields cannot be defined and measured as precisely as in physics and their relationship, in environment–economic interactions as well as in complicated physical systems, may not be known precisely or expected to be simple.

M&G present a version of the formula for intensity of sound measured in decibels (dB) that suits their view, where the logarithm is applied on a dimensionless argument (p. 1605),

$$L_{\mathrm{dB}}=20\log_{10}\left(\frac{P_1}{P_0}\right) \tag{20}$$

where Ps have dimension of power and $P_0=2\times10^{-5}\ \mathrm{N/m^2}$. A quick glance at a compendium of physics formulae (see, e.g., Menzel, 1960) reveals that many equations regularly employed in fields such as thermodynamics, acoustics, chemistry, and astrophysics, include logarithms of dimensional quantities. These formulae include several measures used in environmental sciences such as the moment magnitude scale for earthquakes, $M_w=2/3\log_{10}M_0-10.7$, where M_0 is the magnitude of the seismic moment (10^{-7} Nm), tsunami intensity scales such as, $I=1/2+\log_2 H$, where H is the average wave height on the shore nearest to the source (m), and the measure of the acidity, $pH=-\log_{10}c$, where c is concentration (m^{-3} mol). As Gibbings writes about the intensity of sound equation, these are: "perfectly valid as representations of physical events and as bases of calculation" (Gibbings, 2011, p. 37). All these equation can be rearranged in logarithms of dimensionless arguments.

In contrast, M&G proceeded by "proving" that manipulating logarithmic functions of dimensional quantities is nonsense, by arguing that (p. 1605):

> "it seems that the following expression could be accepted even if a and b were numbers with the same dimension: $\ln a-\ln b=\ln\frac{a}{b}$. But the following formula must hold as well: $\ln a+\ln b=\ln ab$.

[16] This is reminiscent of James Thomas' complaint made in 1878 quoted by (Porter, 1946, p. 6): "a second squared, the square of a second, and the second power of a second of a second of time are all of them essentially meaningless conjunctions of words."

[17] In fact, as shown in Waggoner and Ausubel (2002) and Steinberger and Krausmann (2011), T and A as measured by IPAT are often not analytically independent variables.

[18] For a similar treatment of the production function, see De Jong (1967), De Jong and Kumar (1972) and Szirtes (2006)).

[19] In empirical modeling, these constants become part of the intercept in regressions which are seldom useful mostly because of the problems in estimating them.

If a and b are numbers with the same dimension, three terms in [the second Eq.] do not make any sense. To repeat, in logarithmic formulae, all the terms must be dimensionless pure numbers."

To show the absurdity of their argument we can just assume that a and b have both dimensions of length $[L]$. It is clear that the two equation represent the natural log of two completely different derived physical quantities. Note that, a/b, is a dimensionless quantity $[LL^{-1}]=[1]$, such as an angle, whereas, ab, could be an area $[L^2]$. This means that the first equation will be independent of the units chosen, the second will lead to the introduction of a constant. It is not the intended contradiction that justifies the authors claim.

4.1. Application to Economics and Ecological Economics

M&G commented on the specification by Richmond and Kaufmann (2006, Eq. (8), p. 178)

$$\ln(Y_{it}) = \alpha + \beta\ln(X_{it}) + \phi\ln(\mathbf{Z}_{it}) + \mu_{it},$$

where Y_{it} denotes per capita energy use (BTU/Pop) or per capita carbon emissions (CO_2/Pop); X_{it} denotes per capita GDP; $\mathbf{Z}_{it}$ represents a vector of fuel shares; μ_{it} is the regression error. First, note that there are errors in the equation, which M&G simply reproduce in their own paper.[20] For the sake of simplicity, as it does not alter the main conclusions, let us consider only one variable Z_{it} and omit the error term

$$\ln(Y_{it}) = \alpha + \beta\ln(X_{it}) + \phi\ln(Z_{it}),$$

where Y_{it} has dimension of energy or emission per capita $[EN^{-1}]$, X_{it} has dimension of money per capita $[\$N^{-1}]$, and Z_{it} is dimensionless $[1]$. M&G write that "This form of specification with Y_{it} and X_{it} cannot be accepted." However, it is elementary to show that the equation can be obtained by taking the log of the nonlinear model

$$Y_{it} = \gamma X_{it}^{\beta} Z_{it}^{\phi},$$

where $\gamma = \exp(\alpha)$.[21] This is a (restricted) homogeneous equation. Obviously, we are interested in the slope coefficient, β and ϕ, which are not affected by changes in units. See also Baltagi (2011, p. 71) for a similar example in an basic econometrics textbook.

M&G also compared the following two specification in Arrow et al. (1961, p. 228):

$$\frac{V}{L} = c + dW + \eta \tag{21}$$

$$\log\frac{V}{L} = \log a + b\log W + \epsilon, \tag{22}$$

where V denotes value added in thousands of USD, L is labor input in man-years, and W denotes money wage rate. M&G concluded that relation (22) "cannot be used judging by the dimensions V/L and W which they used." In fact, both equations are homogeneous. It is worth noting that the form of the equation in levels is quite rare as both coefficients are dimensionless since $\frac{V}{L}$ and W have same dimensions of money per labor input. In Eq. (22) the key parameter of interest, the elasticity of substitution between labor and capital is dimensionless. In fact, consider the constant elasticity of substitution (CES) production function

$$V = \left(c_K K^{\rho} + c_L L^{\rho}\right)^{\frac{1}{\rho}}, \tag{23}$$

Table 1
Variables and constants in CES production function.

Variable	Symbol	Dimension
Stock of capital	K	$[R_k]$
Flow of labor	L	$[R_a T^{-1}]$
Flow of goods	V	$[\$T^{-1}]$
Technical parameter L	c_L	$[\$^{\rho} R_a^{-\rho}]$
Technical parameter K	c_K	$[\$^{\rho} R_K^{-\rho} T^{-\rho}]$

with definition of variables and dimensions presented in Table 1. This model is (restricted) homogeneous (see also De Jong and Kumar, 1972). Eq. (22) can be derived using standard economic optimization. Note that the Arrow et al. models in Eqs. (21) and (22) are analogous to the following models with physical justification in the statistical literature as presented in Atkinson (1985, 1994),

$$y^{1/3} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon \tag{24}$$

$$\log y = \beta_0 + \beta_1 \log x_1 + \beta_2 \log x_2 + \epsilon, \tag{25}$$

where y consists of measurements of the volume, x_1 of the diameter and x_2 of the height of 31 cherry trees. In the level model, both sides have the dimension of length $[L]$ whereas the logged model is dimensionless (Atkinson, 1994, p. 307).

M&G also discussed the model by Pindyck (1979),

$$\log P_E = \alpha_0 + \sum_i \alpha_i \log P_i + \sum_i \sum_j \gamma_{ij} \log P_i \log P_j$$

where P_E is an aggregate price index of energy and P_i are the factor prices. Again, M&G concluded "this specification cannot be used for P_i." Two points are important here. First of all, prices are indices, and therefore dimensionless. However, as the choice of dimensions is, as pointed out above, arbitrary, to make dimensional analysis possible, we can assign a dimension to every variable capable of such an assignment.[22] We do so to show that, for all practical purposes, the model is homogeneous. In such cases, we can apply the concept of homogeneous functions directly. We can apply chenges in units of measurement, by multiplying variables by constants, and check whether the equation remains invariant. This is shown in Felipe (1998) where it is also shown that all relevant tests are unaffected by the transformation.

5. Nonhomogeneous Models and Empirical Equations

Physicists themselves admit that there is "more to physics" than what is permitted by dimensional homogeneity. Two related issues have been discussed in the dimensional analysis literature that are important to consider carefully when modeling in ecological economics and related fields: the role of *nonhomogeneous* and *empirical* models.

5.1. Nonhomogeneous Models

Many physicists argue that there is no logical reason why physical laws should be dimensionally homogeneous in the first place (see, e.g., Palacios, 1964, p. 45). In fact, Birkhoff pointed out that if we look at special relativity and quantum mechanics it is obvious that "not all physical laws are dimensionally homogeneous" (Birkhoff, 1960, p. 98). Certainly, there no reasons to assume a priory that a model is dimensionally homogeneous, unless we know that our

[20] There is an undefined vector valued ln function, which is not how it is usually interpreted, and its coefficient is not a vector. The correct approach is to define $\mathbf{Z}_{it}$ as a vector of log-transformed variables, $\boldsymbol{\phi}$ a vector of coefficient with the same size, and use $\boldsymbol{\phi}'\mathbf{Z}_{it}$.

[21] Since $\gamma = Y_{it}/(X_{it}^{\beta} Z_{it}^{\phi})$, the dimension of γ has to be $[E\$^{-\beta}N^{\beta-1}]$.

[22] See De Jong (1967, p. 23–25) for the argument applied to economics. See also Krantz et al. (1971, 482–483) for an analogous discussion on the increased "power" of dimensional analysis if we "choose" to consider angles dimensional quantities, as opposed to the usual view that being the ratio of two lengths, they are dimensionless.

model includes all relevant variables to start with (see Langhaar, 1951, p. 14). Inadequate knowledge of a field cannot be fixed by dimensional analysis (see, e.g., Langhaar, 1951, pp. 14–15). For example, the so called non-Newtonian fluids[23] include constants that vary from substance to substance (see, e.g., Palacios, 1964, pp. 45–46). We must agree with Luce when referring to dimensional invariant types of laws argued that "is adequate for classical physics, but it is too restrictive for modern physics … and probably for other sciences" (see Luce, 1978, p. 15). It is worth reminding ourselves that, if an equation is nonhomogeneous it does not mean that "it does not make any sense" as M&G would like us to believe, by definition, it means that it is valid only in one set of units. In physics you often read that an equation "does not have any physical sense" to capture the latter meaning (see, e.g., Georgescu-Roegen, 1971, p. 402).

5.2. Empirical Equations

In many areas of physics, when an equation is suggested mostly by observations rather than by theory, there is no reason why homogeneity should hold (see, e.g., de St. Q. Isaacson and de St. Q. Isaacson, 1975, p. 37–38). In many other applied sciences, theoretical support can also be limited. According to Lindsey (1999, p. 36) as much as 95% of the work of statisticians is of an exploratory nature (see also Box and Cox, 1964, p. 231).[24] In economics as well, theory can only suggest the importance of a limited set of variables within a particular model. Often there will be several competing models of the same economic phenomenon. The choice of which model performs better, which is the appropriate functional form to use and which variables to include, other than the one suggested by a particular theory, are all considered empirical questions. With potentially every variable endowed with its own dimension, room for dimensional analysis is limited (cf. Bridgman, 1931).

M&G commented on the specification by Morse (2006, p. 82),

$$\log_{10}(\text{GDP/capita}) = 0.772 + 0.0342 \ \text{national IQ},$$

and concluded that "his [log] transformed regression cannot be accepted." Here becomes important to understand the meaning of homogeneity and from where it originates. Do we really believe that IQ is the only determinant of GDP per capita? If not, there is no reason to expect homogeneity. Also, we have to remember that any physical quantity used in dimensional analysis is measured on a ratio scale. Economists are also aware of this basic principle (see, e.g., Allen, 1938, pp. 13–14; De Jong, 1967, pp. 6–12; Georgescu-Roegen, 1971, pp. 97–99). IQ, assuming it measures intelligence, it is an ordinal scale. There is an arbitrary zero,[25] which does not mean absence of intelligence but a zero score on some standardized test. Also, the ratio between two values is not independent of the test used to measure intelligence, so that a country with a score of 90 is not "twice as intelligent" as a country with 45. Since there is no scale invariance of the variables to start with, why should we insist they be part of a dimensionally homogeneous relationship? Failure to understand the meaning of dimensional homogeneity can lead to an incorrect interpretation of the results. It is clear that this estimated relationship cannot be considered a "fundamental law of nature," where a complete set of well defined and precisely measured variables are bound together in an unchanging equation explaining how average national intelligence determines per capita income, but should be regarded as merely a descriptive relationship. As such, it will not be invariant to the units chosen to measure the variables involved and to the sample used to estimate its parameters (see also Hand, 2004, pp. 211–215; Johnson, 1972, pp. 1004–1005; Murphy, 1950, p. 19) In the Kuznets curve literature alone, besides scale ratios, variables that are measured on different scales, such as nominal (geographical dummy variables indicating common borders, linguistic links, and "landlocked" status as in Frankel and Rose, 2005, ordinal (e.g., education, democracy), and interval (e.g., temperature) have been used fruitfully in regression models. As Stevens once said (1946, p. 680): "physical addition, even though it is sometimes possible, is not necessarily the basis of all measurement. Too much measuring goes on where resort can never be had to the process of laying things end-to-end or of piling them up in a heap." It is precisely because of the freedom allowed by empirical models, that ecological economics can explore the relationship between economics and the environment with contributions from several, potentially very different, disciplines.

5.3. Noncomparable R^2

M&G quoted Morse (2006) writing that transforming "the GDP/capita data increases the R^2 from 54% to 70%, suggesting a significant improvement in the model." The reason why the authors argue that "it is worthwhile to investigate the issue of whether or not using logarithmic functions as a dependent variable really improves the least square norm" seems irrelevant here. In fact, ironically, they fail to recognize that it is meaningless to compare models with differently transformed dependent variables, as the two R^2 measure the proportion of explained variations of different variables: levels *GDP/capita*, and transformed log(*GDP/capita*). This elementary observation can be found in most books on statistical regression (see, e.g., Draper and Smith, 1998, p. 246) and basic econometrics (see, e.g., Wooldridge, 2008, p. 192).

6. Conclusion

In this paper we have shown how M&G's conclusions that many papers in economics and ecological economics have serious analytical errors are based on flawed logic, incorrect mathematical proofs, and on a misunderstanding of dimensional homogeneity and the nature of empirical models in applied sciences. If ecological economics wants to promote truly interdisciplinary research it "must go well beyond the fusion of ecology and economics alone. The complex problems of today require a correspondingly complex synthesis of insights and tools from the social sciences, natural sciences, and humanities" (Daly and Farley, 2004, p. xxii). To apply dimensional methods successfully to ecological economics, the traditional perspective of physics needs to be modified to acknowledge fundamental methodological differences in all the relevant contributing disciplines. The authors claim that their criticism is levied in the hope that it "will orient future quantitative analyses toward more constructive ends," however their misguided and narrow application of the dimensional homogeneity principle would actually undermine applied and interdisciplinary research. It is clear that dimensional analysis has to be applied judiciously and requires considerable knowledge of the field of application. Results from nonhomogeneous models are not unacceptable. In fact, they can provide empirical models essential for prediction and decision-making and can be the precursors of future theoretical developments aimed at understanding the surrounding environment.

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[23] Familiar examples of non-Newtonian fluids include chilled caramel topping and Ketchup.
[24] In regression studies, it is sometimes necessary to take an entirely empirical approach to the choice of a relation. In other cases, physical laws, dimensional analysis, etc., may suggest a particular functional form.
[25] So even if amount of difference is assumed to matter, they can only be promoted to an interval scale.

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